# INTERNATIONAL STANDARD

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# Ophthalmic optics and instruments — Reporting aberrations of the human eye

*Optique et instruments ophtalmiques — Méthodes de présentation des aberrations de l'œil humain*

Reference number
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# Foreword

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ISO 24157 was prepared by Technical Committee ISO/TC 172, *Optics and photonics*, Subcommittee SC 7, *Ophthalmic optics and instruments*.

# Ophthalmic optics and instruments — Reporting aberrations of the human eye

## 1 Scope

This International Standard specifies standardized methods for reporting aberrations of the human eye.

## 2 Normative references

The following referenced documents are indispensable for the application of this document. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

ISO 8429, *Optics and optical instruments — Ophthalmology — Graduated dial scale*

## 3 Terms and definitions

For the purposes of this document, the following terms and definitions apply. Symbols used are summarized in Table 1.

**3.1**
**line of sight**
line from the point of interest in object space to the centre of the entrance pupil of the eye and continuing from the centre of the exit pupil to the retinal point of fixation (generally the foveola)

**3.2**
**Zernike polynomial function**
one of a complete set of functions defined and orthogonal over the unit circle, the product of three terms, a normalization term, a radial term and a meridional term, parameterized by a dimensionless radial parameter, $\rho$, and a dimensionless meridional parameter, $\theta$, designated by a non-negative radial integer index, $n$, and a signed meridional index, $m$, and given by the equation

$$Z_n^m = N_n^m R_n^{|m|}(\rho) M(m\theta) \tag{1}$$

where

| | |
|---|---|
| $N_n^m$ | is the normalization term; |
| $R_n^{\vert m\vert}$ | is the radial term; |
| $M(m\theta)$ | is the meridional term; |
| the parameter $\rho$ | is a real number continuous over its range of 0 to 1,0; |
| the parameter $\theta$ | is a real number continuous over its range of 0 to $2\pi$. |

NOTE For a given value of radial index $n$, the meridional index $m$ may only take the values $-n$, $-n+2$, …, $n-2$ and $n$.

**3.2.1**
**radial term**
Zernike polynomial function term with indices $n$ and $m$ given by the equation

$$R_n^{|m|}(\rho) = \sum_{s=0}^{0,5(n-|m|)} \frac{(-1)^s (n-s)!}{s!\left[0,5\left(n+|m|\right)-s\right]!\left[0,5\left(n-|m|\right)-s\right]!} \rho^{n-2s} \quad (2)$$

where $s$ is an integer summation index incremented by one unit

**3.2.2**
**radial parameter**
$\rho$
dimensionless number taking values between 0 and 1, its value at any radial distance, $r$, from the aperture centre being given by the expression

$$\rho = \frac{r}{a} \quad (3)$$

where $a$ is the value of the aperture radius

**3.2.3**
**meridional term**
Zernike polynomial function term with index $m$ given by the equations

$$M(m\theta) = \cos(m\theta) \quad \text{if } m \geqslant 0 \quad (4)$$

$$M(m\theta) = \sin(|m|\theta) \quad \text{if } m < 0 \quad (5)$$

NOTE The meridional term is also known as the *azimuthal* term.

**3.2.4**
**meridional parameter**
$\theta$
angular value taking values between 0 and 2π (0° and 360°), expressed in the coordinate system defined in Clause 4

NOTE This is also called the *azimuthal* angle.

**3.2.5**
**normalization term**
Zernike polynomial function term with indices $n$ and $m$, equal to 1,0 for “un-normalized” functions (3.2.7) and for “normalized” functions (3.2.6) by the equation

$$N_n^m = \sqrt{\left(2-\delta_{0,m}\right)(n+1)} \quad (6)$$

where $\delta_{0,m} = 1$ if $m = 0$, $\delta_{0,m} = 0$ if $m \neq 0$.

**3.2.6**
**normalized Zernike polynomial function**
Zernike polynomial function whose normalization term takes the form given in 3.2.5 for “normalized” functions defined as orthogonal in the sense that it satisfies the following equation

$$\int_0^1 \rho d\rho \int_0^{2\pi} Z_n^m Z_{n'}^{m'} d\theta = \pi \delta_{n,n'} \delta_{m,m'} \quad (7)$$

where

$\delta_{n,n'} = 1$ if $n = n'$, $\delta_{n,n'} = 0$ if $n \neq n'$;

$\delta_{m,m'} = 1$ if $m = m'$, $\delta_{m,m'} = 0$ if $m \neq m'$.

**3.2.7**
**un-normalized Zernike polynomial function**
Zernike polynomial function whose normalization term is equal to 1,0 and defined as orthogonal in the sense that it satisfies the equation

$$\left(2-\delta_{0,m}\right)\left(n+1\right)\int_{0}^{1}\rho d\rho\int_{0}^{2\pi} Z_n^m Z_{n'}^{m'} d\theta = \pi\delta_{n,n'}\delta_{m,m'} \quad (8)$$

where

$\delta_{n,n'} = 1$ if $n = n'$, $\delta_{n,n'} = 0$ if $n \neq n'$;

$\delta_{m,m'} = 1$ if $m = m'$, $\delta_{m,m'} = 0$ if $m \neq m'$;

$\delta_{0,m} = 1$ if $m = 0$, $\delta_{0,m} = 0$ if $m \neq 0$.

**3.2.8**
**order**
value of the radial index $n$ of a Zernike polynomial function

**3.3**
**Zernike coefficient**
member of a set of real numbers, $c_n^m$, which is multiplied by its associated Zernike function to yield a term that is subsequently used in a sum of terms to give a value equal to the best estimate of the surface, $S(\rho,\theta)$, that has been fitted with Zernike terms, such a sum being represented by

$$S\left(\rho,\theta\right) = \sum_{\text{all } n \text{ and } m} c_n^m Z_n^m \quad (9)$$

NOTE 1 Each set of Zernike coefficients has associated with it the aperture diameter that was used to generate the set from surface elevation data. The set is incomplete without this aperture information.

NOTE 2 Annex A gives information on a method to find Zernike coefficients from wavefront slope (gradient) data.

**3.3.1**
**normalized Zernike coefficient**
Zernike coefficient generated using normalized Zernike functions and so designed to be used with them to reconstruct a surface

NOTE Normalized Zernike coefficients have dimensional units of length.

**3.3.2**
**un-normalized Zernike coefficient**
Zernike coefficient generated using un-normalized Zernike functions and so designed to be used with them to reconstruct a surface

NOTE Un-normalized Zernike coefficients have dimensional units of length.

**3.4**
**wavefront error** (of an eye)
$W(x,y)$ **or** $W(r,\theta)$
optical path-length (i.e. physical distance times refractive index) between a plane wavefront in the eye's entrance pupil and the wavefront of light exiting the eye from a point source on the retina, and specified as a function (wavefront error function) of the $(x,y)$ (or $r,\theta$) coordinates of the entrance pupil

NOTE 1 Wavefront error is measured in an axial direction (i.e. parallel to the $z$-axis defined in Clause 4) from the pupil plane towards the wavefront.

NOTE 2 By convention, the wavefront error is set to zero at the pupil centre by subtracting the central value from values at all other pupil locations.

NOTE 3 Wavefront error has physical units of metres (typically reported in micrometres) and pertains to a specified wavelength.

**3.5**
**optical path-length difference**
**OPD**
negative of the **wavefront error** (3.4) at each point in a wavefront representing the correction of the optical path-length needed to correct the wavefront error

**3.6**
**root mean square wavefront error**
**RMS wavefront error**
⟨of an eye⟩ quantity computed as the square root of the variance of the **wavefront error** (3.4) function and defined as

$$RMS_{\text{WFE}} = \sqrt{\frac{\iint\limits_{\text{pupil}} \left[W(x,y)\right]^2 dxdy}{A}} \tag{10}$$

where $A$ is the area of the pupil

or, if the wavefront error function is expressed in terms of normalized Zernike coefficients, a quantity equal to the square root of the sum of the squares of the coefficients with radial indices 2 or greater

$$RMS_{\text{WFE}} = \sqrt{\sum_{n>1,\text{all } m} \left(c_n^m\right)^2} \tag{11}$$

NOTE 1 Piston and average tilt should be excluded from this calculation because they correspond to lateral displacements of the image rather than image degradation *per se*.

NOTE 2 The RMS error can also be found using the discrete set of wavefront error values that were used to generate the Zernike coefficients and standard statistical methods. When this is done it might be found that this RMS value does not exactly match the value found using the formula given above. This is more likely to happen in cases where the locations in the pupil used to sample the wavefront error form a non-uniformly spaced grid. Then the data set does not lead to the formation of discrete, orthogonal Zernike functions.

**3.7**
**higher-order aberrations**
those aberrations experienced by the eye in addition to sphero-cylindrical refractive errors and prismatic error and thus, if the wavefront error is expressed in terms of Zernike polynomial function coefficients, those of order 3 and higher

**3.8**
**wavefront gradient**
$\partial W(x,y)$
vector giving the values of the gradient of the wavefront, $\partial W(x,y)/\partial x$ and $\partial W(x,y)/\partial y$, at locations $x$ and $y$ and, when expressed in terms of Zernike polynomial coefficients, given by:

$$\frac{\partial W(x,y)}{\partial x} = \sum_{\text{all } n \text{ and } m} c_n^m \frac{\partial Z_n^m(x,y)}{\partial x} \quad \text{and} \quad \frac{\partial W(x,y)}{\partial y} = \sum_{\text{all } n \text{ and } m} c_n^m \frac{\partial Z_n^m(x,y)}{\partial y} \tag{12}$$

NOTE Measured gradient values are referred to by $\beta_x(x,y)$ and $\beta_y(x,y)$ at locations $x,y$.

**Table 1 — Symbols**

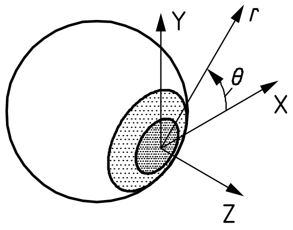
| Symbol | Name | Definition given in |
|---|---|---|
| $A(m\theta,\alpha)$ | meridional term for magnitude/axis Zernike functions | 5.1.9 |
| $c_n^m$ | Zernike coefficient | 3.3 |
| $c_{nm}$ | Zernike coefficient – magnitude | 5.1.9 |
| $m$ | meridional index for Zernike functions | 3.2 |
| $M_n^m(m\theta)$ | meridional term for Zernike functions | 3.2.3 |
| $n$ | radial index for Zernike functions | 3.2 |
| $N_n^m$ | normalization term for Zernike functions | 3.2.5 |
| $R_n^{\|m\|}(\rho)$ | radial term for Zernike functions | 3.2.1 |
| $Z_n^m$ | Zernike function [alternate notation: $Z(n,m)$] | 3.2 |
| $Z_{nm}$ | Zernike function – magnitude/axis form | 5.1.9 |
| $\alpha$ | axis parameter for magnitude/axis form Zernike functions | 5.1.9 |
| $\rho$ | radial parameter for Zernike functions | 3.2.2 |
| $\theta$ | meridional parameter for Zernike functions | 3.2.4 |
| $W(x,y)$ | wavefront error | 3.4 |
| $\beta x,y$ | measured gradient at a location $x,y$ | 3.8 |
| $\partial Wx,y$ | wavefront gradient at a location $x,y$ | 3.8 |
| $\beta_{\text{fit}}$ | gradient fit error | 5.3 |

## 4 Coordinate system

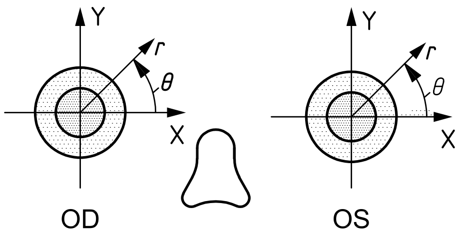
The coordinate system used to represent wavefront surfaces shall be the standard ophthalmic coordinate system in accordance with ISO 8429 in which the $x$-axis is local horizontal with its positive sense to the right as the examiner looks at the eye under measurement, the $y$-axis is local vertical with its positive sense superior with respect to the eye under measurement, the $z$-axis is the line of sight of the eye under measurement with its positive sense in the direction from the eye toward the examiner. The horizontal and vertical origin of the coordinate system is the centre of the visible pupil of the eye. The coordinate system origin lies in the plane of the exit pupil of the eye (for light originating on the retina and passing out through the pupil). This coordinate system is illustrated in Figure 1.

The sign convention used for wavefront error values reported at any location on a wavefront shall be that used for this coordinate system.

When Zernike coefficients are used to represent a wavefront or to report wavefront error, the sign convention used to describe the individual Zernike functions shall be that used for this coordinate system.




a) Coordinate system




b) Clinician's view of patient

**Key**

OD right eye

OS left eye

**Figure 1 — Ophthalmic coordinate system** (ISO 8429)

# 5 Representation of wavefront data

## 5.1 Representation of wavefront data with the use of Zernike polynomial function coefficients

### 5.1.1 Symbols for Zernike polynomial functions

Zernike polynomial functions shall be designated by the upper case letter $Z$ followed by a superscript and a subscript. The superscript shall be a signed integer representing the meridional index of the function, $m$. The subscript shall be a non-negative integer representing the radial index of the function, $n$. Therefore a Zernike polynomial function shall be designated by the form $Z_n^m$ .

If, for reasons of font availability, it is not possible to write superscript and subscripts, the Zernike polynomial functions may be represented as a upper case letter $Z$ followed by parentheses in which the radial index, $n$, appears first, followed, after a comma, by the meridional index, $m$, thus $Z(n,m)$.

### 5.1.2 Radial index

The radial index shall be designated by the lower case letter $n$.

### 5.1.3 Meridional index

The meridional index shall be designated by the lower case letter $m$.

### 5.1.4 Radial parameter

The radial parameter shall be designated by the Greek letter $\rho$.

### 5.1.5 Meridional parameter

The meridional parameter shall be designated by the Greek letter $\theta$.

### 5.1.6 Coefficients

When a surface is represented by Zernike coefficients, these coefficients shall be designated by the lower case letter $c$ followed by a superscript and a subscript. The superscript shall be a signed integer representing the meridional index of the function, $m$. The subscript shall be a non-negative integer representing the radial index of the function, $n$. Therefore, a Zernike coefficient shall be designated by the form $c_n^m$ .

### 5.1.7 Common names of Zernike polynomial functions

Zernike polynomial functions are often referred to by their common names. These names are given in Table 2 in so far as the functions have been given a common name.

**Table 2 — Common names of Zernike polynomial functions**

| Zernike function | Common name |
|---|---|
| $Z_0^0$ | piston |
| $Z_1^{-1}$ | vertical tilt |
| $Z_1^1$ | horizontal tilt |
| $Z_2^{-2}$ | oblique astigmatism |
| $Z_2^0$ | myopic defocus (positive coefficient value)<br>hyperopic defocus (negative coefficient value) |
| $Z_2^2$ | against the rule astigmatism (positive coefficient value)<br>with the rule astigmatism (negative coefficient value) |
| $Z_3^{-3}$ | oblique trefoil |
| $Z_3^{-1}$ | vertical coma – superior steepening (positive coefficient value)<br>vertical coma – inferior steepening (negative coefficient value) |
| $Z_3^1$ | horizontal coma |
| $Z_3^3$ | horizontal trefoil |
| $Z_4^{-4}$ | oblique quatrefoil |
| $Z_4^{-2}$ | oblique secondary astigmatism |
| $Z_4^0$ | spherical aberration<br>positive coefficient value – pupil periphery more myopic than centre<br>negative coefficient value – pupil periphery more hyperopic than centre |
| $Z_4^2$ | with/against the rule secondary astigmatism |
| $Z_4^4$ | quatrefoil |
| $Z_5^{-1}$ | secondary vertical coma |
| $Z_5^1$ | secondary horizontal coma |

### 5.1.8 Comparison of data expressed as Zernike coefficients generated using different aperture sizes

The Zernike coefficient values describing a given wavefront error depend on the aperture size used when they are generated from measurement data. Due to this dependence on pupil diameter, different coefficient values will be found to describe the wavefront error of a given eye if the pupil size changes from one measurement to the next. Therefore, to adequately compare the wavefront error of the same eye at different times or to compare the wavefront errors of two eyes using Zernike coefficients, the compared coefficients shall have been generated using the same pupil diameter even though measurements were taken with different pupil diameters. Zernike coefficients taken at one pupil diameter may be converted into values for a second, smaller pupil diameter using either the method given in Annex B or a similar method.

Wavefront error comparisons using Zernike coefficients found in accordance with this International Standard shall be made between sets of Zernike coefficients that have be converted to a common pupil diameter.

### 5.1.9 Representation of wavefront error data expressed as Zernike coefficients presented in magnitude/axis form

Zernike terms of the same radial order, $n$, and having meridional indices, $m$, with the same magnitude but with opposite signs may be considered to represent the two components of a vector in an angular space with a multiplicity equal to the magnitude of $m$. It is therefore possible to define Zernike functions that combine the functions defined in 3.2 having the same radial order, $n$, and meridional indices with the same magnitude into a new set of functions defined by

$$Z_{nm}(\rho,\theta,\alpha) = N_n^m R_n^{|m|}(\rho) A(m\theta,\alpha) \tag{13}$$

where

$R_n^{|m|}(\rho)$ is defined by 3.2.1;

$N_n^m$ is defined by 3.2.5;

$A(m\theta,\alpha) = \cos\left[m(\theta-\alpha)\right]$

and where $\alpha$ is an angular parameter giving the orientation of the vector in space.

A surface, $S(\rho,\theta)$, such as a wavefront error, is expressed using these Zernike functions as

$$S(\rho,\theta) = \sum_{\text{all } n \text{ and } m} c_{nm} Z_{nm}(\rho,\theta,\alpha_{nm})$$

where the coefficients $c_{nm}$ and the angular parameters $\alpha_{nm}$ are related to the coefficients defined in 3.3 by the equations

$$c_{nm} = \sqrt{\left(c_n^{-m}\right)^2 + \left(c_n^m\right)^2} \tag{14}$$

$$\alpha_{nm} = \frac{a\tan\left(\frac{c_n^{-m}}{c_n^m}\right)}{|m|} \tag{15}$$

### 5.1.10 Common names of Zernike polynomial functions – magnitude/axis form

Zernike polynomial functions are often referred to by their common names. For the magnitude/axis Zernike functions defined in 5.1.9, these names are given in Table 3 in so far as the functions have been given a common name.

**Table 3 — Common names of Zernike polynomial functions – magnitude/axis form**

| Zernike function | Common name |
|---|---|
| $Z_{00}$ | piston |
| $Z_{11}$ | tilt |
| $Z_{20}$ | myopic defocus (positive coefficient value)<br>hyperopic defocus (negative coefficient value) |
| $Z_{22}$ | astigmatism<br>against the rule, axis = 180°<br>with the rule, axis = 90° |
| $Z_{31}$ | coma |
| $Z_{33}$ | trefoil |
| $Z_{40}$ | spherical aberration<br>positive coefficient value – pupil periphery more myopic than centre<br>negative coefficient value – pupil periphery more hyperopic than centre |
| $Z_{42}$ | secondary astigmatism |
| $Z_{44}$ | quatrefoil |
| $Z_{51}$ | secondary coma |

## 5.2 Representation of wavefront data in the form of wavefront gradient fields or wavefront error function values

### 5.2.1 Gradient values

The measurements made of the aberrations of the eye by aberrometers are in general measurements of the gradient of the wavefront error function. Measurements of this type may also be thought of as measurements of the deflection of rays from an un-aberrated direction by the optical system of the eye. In the case of rays originating at the retina and measured as they pass the exit pupil, the deflection is measured from the ray to a ray at the same pupil location but parallel to the line of sight. In the case of rays entering the eye through its entrance pupil, the deflection is measured from the ray to a ray that enters the eye parallel to the line of sight and is refracted so that it intersects the retina at the point the line of sight intersects the retina. The gradient information consists of the two-dimensional location of the measured ray in the plane of the exit pupil of the eye and the two components of its deflection.

So that this information may be conveyed in a standardized fashion, the data for each measured ray or location in the wavefront will consist of four numbers. The first two are the horizontal ($x$) and vertical ($y$) coordinates of the location given as Cartesian coordinates in the coordinate system specified in Clause 4 and expressed in millimetres. The second two numbers are the horizontal and vertical component values of the gradient or, to state this another way, the second two numbers are the horizontal and vertical deflections of the ray given as tangent values.

A fifth, optional, number may be included giving the quality or certainty associated with the information given at each data location.

### 5.2.2 Wavefront error values

If the aberrations of the eye consist of the values of the wavefront error function itself, then the information needed to express this at a given location in the wavefront consists of the location and the value of the wavefront error functions at that location.

So that this information may be conveyed in a standardized fashion, the data for each measured ray or location in the wavefront will consist of three numbers. The first two are the horizontal ($x$) and vertical ($y$) coordinates of the location given as Cartesian coordinates in the coordinate system specified in Clause 4 and expressed in millimetres. The third number is the value of the wavefront error (3.4) expressed in micrometres.

A fourth, optional, number may be included giving the quality or certainty associated with the information given at each data location.

## 5.3 Gradient fit error

When the occasion arises that a wavefront error function has been reconstructed from measured wavefront gradient data and the values for the reconstructed wavefront are conveyed either in the form of gradient fields in accordance with 5.1 or the wavefront error function itself in accordance with 5.2, this information shall be accompanied by two additional values for each data location that give information on the quality of the fit of the data to the reconstruction. The first such value is the difference between the measured $x$ gradient value and the reconstructed $x$ gradient value. The second value is the difference between the measured $y$ gradient value and reconstructed $y$ gradient value. These two values are defined as the gradient fit error and constitute the two components of an error gradient field. These two values are not to be taken as the same as the optional quality values allowed for in 5.1 and 5.2 as those values refer to the quality of the measured data themselves whereas the gradient fit values refer to the quality of the fit of the data to the reconstructed wavefront.

The gradient fit parameter $\beta_{\text{fit}}$ is a metric measure that can be used to identify the overall quality of the fit. It is generally the merit function that is minimized in least-squares fitting. It is defined by:

$$\beta_{\text{fit}} = \sqrt{\frac{\sum_{x,y}\left[\beta_x(x,y) - \frac{\partial W(x,y)}{\partial x}\right]^2}{N} + \frac{\sum_{x,y}\left[\beta_y(x,y) - \frac{\partial W(x,y)}{\partial y}\right]^2}{N}} \quad (16)$$

The various parameters are defined in 3.8.

# 6 Presentation of data representing the aberrations of the human eye

## 6.1 General

The preferred method of *communicating* aberration data for the human eye to others so that they may analyse them as they see them is in the form of a set of gradient components for each measured location as specified in 5.2.1. These values fully characterize the measured wavefront error of the eye and may be used to reconstruct that wavefront error surface using any method desired.

However, wavefront data in the form of gradient components do not convey the wavefront information in a fashion that is easily understood nor in a form that lends itself for convenient use in papers, displays and other common forms of communication. Thus the preferred methods for *presenting* aberration data of the human eye are:

a) as a list of normalized Zernike coefficients;

b) as a bar chart showing the values of the normalized Zernike coefficients;

c) in the form of a topographical map of the wavefront surface.

## 6.2 Aberration data presented in the form of normalized Zernike coefficients

### 6.2.1 Aperture information

When data representing the aberrations of the human eye are presented in the form of normalized Zernike coefficients, the aperture diameter used in generating the coefficient shall form a part of the data set and shall be the first member of the set.

### 6.2.2 Units

Coefficients shall be given in units of micrometres. Aperture size shall be given in millimetres.

### 6.2.3 Ordering of terms

When data representing the aberrations of the human eye are presented in the form of Zernike coefficients, the coefficients shall be listed in the following order.

The first value of the coefficient set shall be the value of the aperture diameter used to generate the set, followed by the Zernike coefficients grouped by common radial index, $n$, with these groups listed in increasing magnitude of the radial index. Within a group of common radial indices, the coefficients shall be listed in increasing value of the meridional index, $m$, starting with the most negative $m$ value and proceeding in increasing order to the most positive $m$ value. The following single index $j$ gives the above ordering for the Zernike coefficients. The first Zernike coefficient of the set has the ordering index value zero.

$$j = \frac{n(n+2)+m}{2} \tag{17}$$

### 6.2.4 Form of presentation

#### 6.2.4.1 Tabular form

When the normalized Zernike coefficients are presented as a table the first column shall contain the Zernike function symbols ordered in accordance with 6.2.3. The second column shall contain the numerical values of the coefficients, aligned with their respective function symbols. If it is desired to present names for the Zernike functions, these names may be placed in a third column. The first row of the table shall contain the words “aperture diameter” in the first column and the aperture diameter value in the second column.

#### 6.2.4.2 Bar chart form

When the normalized Zernike coefficients are presented as a bar chart the values assigned to the bars shall be values of the normalized Zernike coefficients ordered in accordance with 6.2.3. The bars shall be labelled using Zernike double index symbols in accordance with 5.1.1. The values of the bars shall be given in micrometres. The value of the aperture diameter used to create the coefficient values shall appear within the graph.

## 6.3 Aberration data presented in the form of normalized Zernike coefficients given in magnitude/axis form

### 6.3.1 Aperture information

When data representing the aberrations of the human eye are presented in the form of normalized Zernike coefficients, the aperture diameter used in generating the coefficient shall form a part of the data set and shall be the first member of the set.

#### 6.3.2 Units

Coefficients shall be given in units of micrometres. Aperture size shall be given in millimetres. Axes shall be given in degrees.

#### 6.3.3 Ordering of terms

When data representing the aberrations of the human eye are presented in the form of Zernike coefficients, given in magnitude/axis form in accordance with 5.1.9, the coefficients, $c_{nm}$ and axis values, $\alpha_{nm}$, shall be listed in the following order.

The first value of the coefficient set shall be the value of the aperture diameter used to generate the set followed by the coefficient magnitudes and axes grouped by common radial index, $n$, with these groups listed in increasing magnitude of the radial index. Within a group of common radial indices, the coefficients shall be listed in increasing value of the meridional index, $m$.

#### 6.3.4 Tabular form of presentation

When the normalized Zernike coefficients are presented as a table the first column shall contain the Zernike function symbols ordered in accordance with 6.3.3. The second column shall contain the numerical values of the coefficients, aligned with their respective function symbols. The third column shall contain the numerical values of the axes, aligned with their respective function symbols. If it is desired to present names for the Zernike functions, these names may be placed in a fourth column. The first row of the table shall contain the words "aperture diameter" in the first column and the aperture diameter value in the second column.

### 6.4 Aberration data presented in the form of topographical maps

#### 6.4.1 General

In order to facilitate the interpretation and comparison of wavefront measurements from different systems, criteria for a standard graphical display of ocular wavefront aberrations are established. The elements of the display are: colour set, aberration scale, colour contour map, numeric data, spatial scale and title. If compliance with this International Standard is claimed, this standardized display of ocular wavefront aberrations shall be made available to the user and shall contain the text "ISO 24157".

#### 6.4.2 Display contents

##### 6.4.2.1 Standardized display

This shall contain the following elements:

— title;

— colour legend graphic;

— step size text;

— colour contour map of higher-order aberrations (wavelength 0,555 µm, if possible);

— numeric data;

— spatial extent indicator;

— axis indication;

— reference to ISO 24157.

#### 6.4.2.2 Display title

The display shall be titled “Higher order aberrations”.

#### 6.4.2.3 Colour contour map

The colour map shows a colour-coded representation of the higher-order aberrations in the entrance pupil of the eye. If possible, these aberrations should be referenced to wavelength 0,555 µm. The centre value of the map shall be zero corresponding to the chief ray.

#### 6.4.2.4 Numeric data

The numeric data to be displayed on the map shall include:

— low-order aberrations' RMS value, in micrometres, computed for aberrations second-order;

— higher-order aberrations' RMS value, in micrometres, computed for aberrations third-order and above;

— total aberrations' RMS value, in micrometres, computed for aberrations second-order and above;

— diameter of pupil, in millimetres.

#### 6.4.2.5 Spatial extent graphic

The spatial extent graphic shows the size of the colour map. It shall consist of some graphic indicator and text indicating the width of the display in units of millimetres.

#### 6.4.2.6 Axis indicator

The axis indicator graphic shows the angular coordinate system as defined in Clause 4.

### 6.4.3 Standardized scales

For display of the higher-order aberrations, the standardized display shall use one of four step size intervals:

0,1 µm, 0,2 µm, 0,5 µm, and 1,0 µm

The step size shall be prominently displayed below the colour legend. Twenty-one (21) colours shall be used with the centre colour set at zero. If the aberration value to be displayed exceeds the range of the colour scale, the highest or lowest colour (as appropriate) shall be used.

### 6.4.4 Colour palette

The twenty-one colours shall follow the general guidelines:

— where the wavefront error as defined in 3.4 takes negative values, cooler colours are used (blues);

— where the wavefront error as defined in 3.4 takes positive values, warmer colours are used (reds);

— where the wavefront error as defined in 3.4 is zero, green is used.

## 6.5 Presentation of pooled aberration data

### 6.5.1 General

For multiple eye studies of the aberrations of the eye it is desirable to present pooled results. When this is done, certain precautions need to be taken both in the analysis and presentation of the data if the pooled results are to be meaningful. In sets of data from multiple sources and from many individuals it is almost certain that not all the data will come from eyes having the same pupil size. Therefore steps shall be taken to account for changes in aberration values found when the pupil size changes. It is also quite likely that both right and left eyes are included in a study. When this is the case this fact needs to be acknowledged along with any steps that have been taken to account for known anatomical asymmetries that occur between right and left eyes.

### 6.5.2 Analysis and presentation of pooled aberration results based on Zernike coefficient sets

When the data used for a study is in the form of Zernike polynomial coefficients, certain precautions need to be taken both in the processing and analysis of the data and in its presentation. This is because the values of the Zernike coefficients that describe a given wavefront will change, even if they are given in accordance with this International Standard, if the aperture diameter used is changed. Therefore the coefficients may not be directly compared until all use a common pupil diameter. For this reason the first step in the analysis of pooled data sets is to convert all Zernike coefficient sets to sets having the same pupil size. This may be done using the method given in Clause B.2.

If the data are given in the form specified in 5.2.1 or 5.2.2 but the pooled results are presented in the form of Zernike coefficients, the coefficients shall be generated from the data using the same pupil diameter for all eyes.

### 6.5.3 Analysis and presentation of pooled aberration data where both right and left eye data are used

When both right eye and left eye data are used in pooled data sets, consideration needs to be taken for the known asymmetries about the vertical meridian of the eye between right and left eyes. This may be done in one of two ways.

If aberration data include measurements from both eyes, which have not been altered to compensate for known asymmetries, it is preferable to analyse the data and present the results separately for the right eyes and the left eyes. If it is decided to pool data from both right and left eyes in the same analysis, this fact shall be stated explicitly.

If it is decided to pool data from both right and left eyes in the same analysis and when analysis and presentation is done based on Zernike coefficient sets, the known anatomical asymmetry may be accounted for by altering the sign of all Zernike coefficients that arise from Zernike polynomials with negative, even meridional indices, and with positive, odd meridional indices for all left eye data prior to analysis. This step has the effect of giving left eyes the same asymmetry, on average, as that found in right eyes.

If the data are given in the form of either gradient arrays in accordance with 5.2.1 or elevation arrays in accordance with 5.2.2, the right/left asymmetry may be accounted for by changing the sign of all $x$ location values for the left eyes prior to analysis of the data. This step has the effect of giving left eyes the same asymmetry, on average, as that found in right eyes.

When either of the above steps has been taken to account for the asymmetries between right and left eyes, this fact shall be prominently stated.

# Annex A
(informative)

# Methods of generating Zernike coefficients

In general, aberrometry measures the slope (surface gradient) of the wavefront in the eye. The Shack-Hartmann and Spatially-Resolved Refractometry techniques measure wavefront slope directly and the Tscherning and retinal raytracing techniques measure the transverse ray error, which is proportional to the wavefront slope. The slope measurements are the local tilt of the wavefront and can be in both the horizontal and vertical directions. For a wavefront error given by $W(x,y)$, aberrometers give a set of slope measurements $\{dW(x_i,y_i)/dx\}$ and $\{dW(x_i,y_i)/dy\}$. The value of $i$ ranges from 1 to the total number of sample points, $N$. The points $(x_i,y_i)$ represent the locations of the individual samples. The most common method of reconstructing the wavefront error based on this slope information is fitting the data to a set of polynomials $\{V_j\}$ with a least squares technique, where $j$ ranges from unity to the total number of polynomials in the fitting set, $J$. Typically, the polynomial set for wavefront fitting has been either the Zernike polynomials or the Taylor polynomials. These sets have traditionally been chosen because they have properties that represent familiar concepts in ophthalmic optics. However, other polynomial sets can also be used. The least squares technique minimizes the absolute error between the measured wavefront gradients and the derivatives of the reconstructed wavefront. A typical merit function is given in 5.3 ($\beta_{\text{fit}}$). To perform this fit, a matrix equation is set up, such that

$$\left[\boldsymbol{dV}\right]\vec{\boldsymbol{a}} = \boldsymbol{d}\vec{\boldsymbol{W}} \tag{A.1}$$

where

$$\left[\boldsymbol{dV}\right] = \begin{bmatrix} dV_1(x_1,y_1)/dx & dV_2(x_1,y_1)/dx & \cdots & dV_J(x_1,y_1)/dx \\ dV_1(x_2,y_2)/dx & dV_2(x_2,y_2)/dx & \cdots & dV_J(x_2,y_2)/dx \\ \vdots & \vdots & \vdots & \vdots \\ dV_1(x_N,y_N)/dx & dV_2(x_N,y_N)/dx & \cdots & dV_J(x_N,y_N)/dx \\ dV_1(x_1,y_1)/dy & dV_2(x_1,y_1)/dy & \cdots & dV_J(x_1,y_1)/dy \\ dV_1(x_2,y_2)/dy & dV_2(x_2,y_2)/dy & \cdots & dV_J(x_2,y_2)/dy \\ \vdots & \vdots & \vdots & \vdots \\ dV_1(x_N,y_N)/dy & dV_2(x_N,y_N)/dy & \cdots & dV_J(x_N,y_N)/dy \end{bmatrix} \tag{A.2}$$

$$\vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_J \end{pmatrix} \tag{A.3}$$

and

$$\boldsymbol{d}\vec{\boldsymbol{W}} = \begin{bmatrix} dW(x_1,y_1)/dx \\ dW(x_2,y_2)/dx \\ \vdots \\ dW(x_N,y_N)/dx \\ dW(x_1,y_1)/dy \\ dW(x_2,y_2)/dy \\ \vdots \\ dW(x_N,y_N)/dy \end{bmatrix} \tag{A.4}$$

Essentially, the $\left[\boldsymbol{dV}\right]$ matrix contains slope information for the fitting polynomial functions. The top half of the matrix contains the $x$ derivatives of the fitting function and the lower half contains the $y$ derivatives of the fitting functions. Each row in the $\left[\boldsymbol{dV}\right]$ matrix is for a given sample point $(x_i,y_i)$. Each column in this matrix is for a different fitting polynomial function $V_j$. The $\vec{\boldsymbol{a}}$ vector is a series of weighting coefficients which describes how much of each fitting polynomial function contributes to the reconstructed wavefront. The $\boldsymbol{d\vec{W}}$ vector contains the data measured from the aberrometer. The upper half of this matrix contains the $x$ derivative information, while the lower half of the matrix describes the $y$ derivative data. Each row in the $\boldsymbol{d\vec{W}}$ vector is for a different sample point $(x_i,y_i)$. The goal of the reconstruction is to determine the values of the coefficients in the $\vec{\boldsymbol{a}}$ vector. Since there are usually many more sample points, $N$, than there are polynomial functions, $J$, to fit the wavefront, an exact solution to Equation (A.1) cannot be obtained. Instead, a least squares solution is calculated. The solution is given by

$$\vec{\boldsymbol{a}} = \left[\left[\boldsymbol{dV}\right]^{\boldsymbol{T}}\left[\boldsymbol{dV}\right]\right]^{-1}\left[\boldsymbol{dV}\right]^{\boldsymbol{T}}\boldsymbol{d\vec{W}} \tag{A.5}$$

where $\left[\boldsymbol{dV}\right]^{\boldsymbol{T}}$ is the transpose of the matrix $\left[\boldsymbol{dV}\right]$, and $\left[\left[\boldsymbol{dV}\right]^{\boldsymbol{T}}\left[\boldsymbol{dV}\right]\right]^{-1}$ is the inverse matrix operation. Equation (A.5) allows for the calculation of the fitting coefficients. The final reconstructed wavefront $W(x,y)$ is then given by

$$W(x,y) = \sum_{j=1}^{J} a_j V_j(x,y) \tag{A.6}$$

Thus, once the slope information is obtained from the aberrometer, straightforward matrix calculations are all that is required to reconstruct the wavefront.

A final word needs to be said about giving the coefficient set $\vec{\boldsymbol{a}}$ the correct dimensions and scaling. Both $\boldsymbol{d\vec{W}}$, the measured gradient values, and the elements of $\left[\boldsymbol{dV}\right]$, the gradient components of the various members of the polynomial function set used, evaluated at the measurement locations are dimensionless numbers whereas the values in $\vec{\boldsymbol{a}}$ shall have dimensions of length. This is so that when they are multiplied by the various polynomial values for a given location, the resulting sum is a height value with dimensions of length. If Zernike functions constitute the polynomial set, correct scaling will occur if the coefficient set $\vec{\boldsymbol{a}}$ found using Equation (A.5) is multiplied by the aperture radius used in the reconstruction thereby giving the correct coefficient set, $\vec{\boldsymbol{a}}$, to be used in Equation (A.6).

# Annex B
(informative)

# Conversion of Zernike coefficients to account for differing aperture sizes, decentration and coordinate system rotation

## B.1 General

This annex gives information on methods for converting sets of Zernike coefficients generated using a coordinate system with a given origin and rotational orientation to another having a second coordinate origin and/or rotational orientation. It also gives information on methods for converting a set of Zernike coefficients representing a wavefront in an area within a first aperture diameter to a set of Zernike coefficients representing the same wavefront but in an area within a second diameter. When all available information on wavefront error is contained in sets of Zernike coefficients, these methods must be used if the aberrations are to be compared to one another or if statistical analysis of populations is to be done. These methods also provide analytical tools to study the optical/visual effects of misplaced corrections to the aberrations of the eye and to study the optical/visual effects of changes in pupil size in the presence of aberrations of the eye.

If full information is available on the aberrations of the eye in the form of complete sets of measured data such as those specified in 5.2, it is best to account for the effects of decentration, rotation and resizing of the pupil by using selected portions of the complete data set and reconstructing the wavefront and its effects from the full data. This is because Zernike coefficient sets represent a fit of the basic data to a limited set of basic functions and so may be thought of as a form of data compression in which some information is inevitably lost. If the best fit to the original data, represented by an original coefficient set, does not fit selected portions of the data field well, it may well be found that selecting the portion of the data field that is of interest and refitting will give more reliable results than will mathematically transforming the originally created set of Zernike coefficients using the methods given in this annex.

## B.2 Conversion of Zernike coefficients to account for differing aperture sizes

In general, aberrometer measurements will have different pupil sizes. When wavefronts are calculated based on these data, they are usually represented in terms of Zernike polynomials. Zernike polynomials are normalized to have a unit radius. When comparing two distinct wavefronts, the pupil size may not be the same. Consequently, the Zernike expansion coefficients cannot be directly compared because of differences in pupil normalization. To make a comparison, it is desirable to recalculate the wavefront measurement with the larger pupil size to fit over the pupil of the wavefront with the smaller pupil size. However, for gradient data that have a poor wavefront fit (as identified by a large gradient fit parameter $\beta_{\text{fit}}$ as described in 5.3), this can lead to significant errors. Care should be taken to consider this parameter prior to recalculating Zernike coefficients for a smaller diameter. If available, the original gradient data should be re-fit over the smaller pupil to avoid this potential problem.

There are a variety of ways to scale Zernike coefficients to a different pupil size. The following method is given as it allows convenient algorithm implementation.

First, the ratio $\eta$ is formed by dividing the new aperture size by the original aperture size. Next diagonal matrix [$\eta$] is formed from powers of $\eta$ in the following fashion. First let the terms of a set of Zernike functions, through order $n_{\mathrm{max}}$, the maximum value that the radial index $n$ takes for the set, be ordered in the following non-standard fashion. The terms are first grouped by meridional index, $m$, starting from the most negative value allowed by the set, $m = -n_{\mathrm{max}}$, and progressing in the positive sense to the most positive value allowed by the set, $m = n_{\mathrm{max}}$. Then, within each group of meridional index $m$ the functions are ordered by radial index starting with the smallest allowed value, $n = |m|$, and proceeding to the highest value allowed by the entire set. With a group of meridional index $m$, the allowed values of $n$ must have the same parity as $m$, odd or even. If the parity of $m$ is the same as $n_{\mathrm{max}}$, then the maximum radial index value for the group is $n_{\mathrm{max}}$. Otherwise the maximum radial value is $n_{\mathrm{max}}-1$. The Zernike functions, ordered in this fashion, are now illustrated by a functional row vector

$$\langle Z| = \left[ Z_{n_{\mathrm{max}}}^{-n_{\mathrm{max}}},\ \ Z_{n_{\mathrm{max}}-1}^{-(n_{\mathrm{max}}-1)},\ \ Z_{n_{\mathrm{max}}-2}^{-(n_{\mathrm{max}}-2)},\ \ Z_{n_{\mathrm{max}}}^{-(n_{\mathrm{max}}-2)},\ \ \cdots\ \ Z_{n_{\mathrm{max}}}^{(n_{\mathrm{max}}-2)},\ \ Z_{n_{\mathrm{max}}-1}^{n_{\mathrm{max}}-1},\ \ Z_{n_{\mathrm{max}}}^{n_{\mathrm{max}}} \right] \tag{B.1}$$

Starting from the upper left diagonal element of [$\eta$], as the first element, and proceeding to the lower right diagonal element, insert values equal to $\eta$ raised to the power of the radial order index of the corresponding element of $\langle \boldsymbol{Z}|$, proceeding from left to right through $\langle \boldsymbol{Z}|$.

Next, form block diagonal matrix [$\boldsymbol{R}$] whose block elements, [$\boldsymbol{R}(n,m)$], are each associated with a single meridional index $m$ and whose columns are associated with the radial index $n$ that matches the radial index value of the corresponding diagonal position in [$\eta$]. The elements of each column, from top to bottom, are the weighting coefficients given by 3.2.1 to radial term $R_n^{|m|}(\rho)$.

Next, form diagonal matrix [$\boldsymbol{N}$] by inserting the value of $N_n^m$ given by 3.2.4 for the value of $n$ and $m$ for the same location in matrix [$\boldsymbol{R}$].

Finally, form permutation matrix [$\boldsymbol{P}$] and re-order the original coefficient set in the following fashion. The entire set of ordered original Zernike coefficients is represented by a vector of coefficient values, symbolized by

$$|c\rangle = \begin{pmatrix} c_0^0 \\ c_1^{-1} \\ \vdots \\ c_n^m \\ \vdots \end{pmatrix} \tag{B.2}$$

Square matrix [$\boldsymbol{P}$] with the same number of rows and columns as the column vector $|\boldsymbol{c}\rangle$ has its elements first filled completely with zeros. Then in each row of that matrix one element is changed to a one (1). Each row in [$\boldsymbol{P}$] is associated with the same row in $|\boldsymbol{c}\rangle'$, the re-ordered vector of coefficient values. In a given row, the 1 is placed in the column whose index value is the same as the row value of the element of $|\boldsymbol{c}\rangle$ that is to be placed in the selected row of $|\boldsymbol{c}\rangle'$.

The matrices thus formed are combined, along with their inverses, to form the conversion matrix [$\boldsymbol{C}$], having the form

$$[\boldsymbol{C}] = [\boldsymbol{P}]^{\mathrm{T}}[\boldsymbol{N}]^{-1}[\boldsymbol{R}]^{-1}[\eta][\boldsymbol{R}][\boldsymbol{N}][\boldsymbol{P}] \tag{B.3}$$

and the resized coefficients, $|\boldsymbol{c}\rangle'$, are formed from the original ones, $|\boldsymbol{c}\rangle$, using the matrix equation

$$|\boldsymbol{c}\rangle' = [\boldsymbol{C}]|\boldsymbol{c}\rangle \tag{B.4}$$

## B.3 Conversion of Zernike coefficients to account for decentration of the coordinate origin

If the same wavefront data is analysed using Zernike decomposition, first with one coordinate origin selection and then with a second coordinate origin selection, the coefficients obtained will be different. Therefore care must be taken when comparing sets of Zernike coefficients, or using them for some purpose with a predefined coordinate origin, that the same coordinate system is used throughout. If the amount of coordinate system translation (decentration) from the preferred coordinate system is known, preferably in terms of $x$ and $y$ decentration values, the Zernike coefficients may be transformed to the preferred system using the following method.

Any analytic function such as a surface expressed as a sum of weighed Zernike polynomials in accordance with 3.3, that has a value at location $(x,y)$ of $S(x,y)$ and that has partial derivatives of all orders, may be evaluated at location $(x+dx, y+dy)$ using the Taylor expansion

$$S(x+dx, y+dy) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right)^k S(x,y) \tag{B.5}$$

Any function, such as $S(x,y)$, may also be expanded using a complete set of orthogonal functions, $Z_n^m$ $(x,y)$, with suitable weighting coefficients, $c_n^m$, and be represented as

$$S(x,y) = \sum_{\text{all } n \text{ and } m} c_n^m Z_n^m (x,y) \tag{B.6}$$

By inserting (B.6) into (B.5) it is found that

$$S(x+dx, y+dy) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right)^k \sum_{\text{all } n \text{ and } m} c_n^m Z_n^m (x,y)$$

Since $c_n^m$ is not a function of $x$ or $y$ it is not effected by the action of the partial differential operators and so the expression may be rearranged to read

$$S(x+dx, y+dy) = \sum_{\text{all } n \text{ and } m} c_n^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right)^k Z_n^m (x,y) \tag{B.7}$$

Use of Equation (B.7) generally requires the calculation and evaluation of partial derivatives of all orders for all members of the orthogonal set. However for a special class of orthogonal functions whose first partial derivatives may be expressed as sums of the original set itself, the evaluation of Equation (B.7) may be greatly simplified as will now be shown. The Zernike polynomial functions represent one such set of this special class of orthogonal functions, as was first shown by Noll (see Bibliography reference [3]).

The following equations give the relationships between the un-normalized Zernike functions and their first derivatives.

$$\frac{\partial}{\partial x} Z_n^m = (1+\delta_{m0}) \left[ \sum_{n'=|m|+1}^{n-1} (n'+1) Z_{n'}^{\frac{m}{|m|}(|m|+1)} + (1-\delta_{m0})(1-\delta_{m,-1}) \sum_{n'=|m|-1}^{n-1} (n'+1) Z_{n'}^{\frac{m}{|m|}(|m|-1)} \right] \tag{B.8}$$

$$\frac{\partial}{\partial y} Z_n^m = \left(1+\delta_{m0}\right)\frac{m}{|m|}\left[\sum_{n'=|m|+1}^{n-1}\left(n'+1\right)Z_{n'}^{-\frac{m}{|m|}\left(|m|+1\right)} - \left(1-\delta_{m0}\right)\left(1-\delta_{m1}\right)\sum_{n'=|m|-1}^{n-1}\left(n'+1\right)Z_{n'}^{-\frac{m}{|m|}\left(|m|-1\right)}\right] \quad \text{(B.9)}$$

where $\delta_{ab} = 1$ if $a = b$; $\delta_{ab} = 0$ if $a \neq b$ and $n'$ is incremented by 2 in the summations.

Note that if ($|m|$+1) is larger than ($n$–1), the first sum in (B.8) and (B.9) is non-existent because the structure of Zernike polynomial functions is such that $n$ must always be greater than or equal to $|m|$.

Let the first partial derivatives of an un-normalized Zernike function be given by the following sums

$$\frac{\partial Z_n^m(x,y)}{\partial x} = \sum_{j=1}^{\infty} Dx_{ij} Z_n^m(x,y) \quad \text{(B.10)}$$

$$\frac{\partial Z_n^m(x,y)}{\partial y} = \sum_{j=1}^{\infty} Dy_{ij} Z_n^m(x,y) \quad \text{(B.11)}$$

where the $Dx_{ij}$ and $Dy_{ij}$ are weighting coefficients, $i$ is the single index designation for the index pair ($n$,$m$) of the Zernike function whose derivative is being found and $j$ is the single index designation for the entire Zernike function set in accordance with 6.2.3.

By representing the first partial derivatives of all members of the Zernike function set by the column vectors

$$\left|\partial \boldsymbol{Zx}\right\rangle = \begin{pmatrix} \frac{\partial}{\partial x} Z_0^0 \\ \frac{\partial}{\partial x} Z_1^{-1} \\ \vdots \\ \frac{\partial}{\partial x} Z_n^m \\ \vdots \end{pmatrix} \quad \text{and} \quad \left|\partial \boldsymbol{Zy}\right\rangle = \begin{pmatrix} \frac{\partial}{\partial y} Z_0^0 \\ \frac{\partial}{\partial y} Z_1^{-1} \\ \vdots \\ \frac{\partial}{\partial y} Z_n^m \\ \vdots \end{pmatrix}$$

the members of the Zernike function set itself by the column vector

$$\left|\boldsymbol{Z}\right\rangle = \begin{pmatrix} Z_0^0 \\ Z_1^{-1} \\ \vdots \\ Z_n^m \\ \vdots \end{pmatrix}$$

and the weighting coefficients by two matrices

$$\left[\boldsymbol{Dx}\right] = \begin{bmatrix} Dx_{11} & Dx_{12} & \cdots & Dx_{1j} & \cdots \\ Dx_{21} & Dx_{22} & \cdots & Dx_{2j} & \cdots \\ \vdots & \vdots & \ddots & & \\ Dx_{j1} & Dx_{j2} & & Dx_{jj} & \\ \vdots & \vdots & & & \ddots \end{bmatrix} \quad \text{and} \quad \left[\boldsymbol{Dy}\right] = \begin{bmatrix} Dy_{11} & Dy_{12} & \cdots & Dy_{1j} & \cdots \\ Dy_{21} & Dy_{22} & \cdots & Dy_{2j} & \cdots \\ \vdots & \vdots & \ddots & & \\ Dy_{j1} & Dy_{j2} & & Dy_{jj} & \\ \vdots & \vdots & & & \ddots \end{bmatrix}$$

the ensemble of equations given by Equations (B.10) and (B.11) may be represented by the compact matrix equations

$$\left|\partial \boldsymbol{Zx}\right\rangle = \left[\boldsymbol{Dx}\right]\left|\boldsymbol{Z}\right\rangle \tag{B.12}$$

$$\left|\partial \boldsymbol{Zy}\right\rangle = \left[\boldsymbol{Dy}\right]\left|\boldsymbol{Z}\right\rangle$$

Now consider the second partial derivatives of the special orthogonal functions. For instance, using the formalism of Equations (B.12), we may write for the second partial derivatives with respect to $x$ as

$$\frac{\partial}{\partial x}\left|\partial \boldsymbol{Zx}\right\rangle = \frac{\partial}{\partial x}\left[\boldsymbol{Dx}\right]\left|\boldsymbol{Z}\right\rangle$$

Since none of the elements of the coefficient matrix [$\boldsymbol{Dx}$] is a function of $x$, this expression becomes

$$\frac{\partial}{\partial x}\left|\partial \boldsymbol{Zx}\right\rangle = \left[\boldsymbol{Dx}\right]\left(\frac{\partial}{\partial x}\left|\boldsymbol{Z}\right\rangle\right) = \left[\boldsymbol{Dx}\right]\left|\partial \boldsymbol{Zx}\right\rangle = \left[\boldsymbol{Dx}\right]\left[\boldsymbol{Dx}\right]\left|\boldsymbol{Z}\right\rangle$$

Defining the notation

$$\frac{\partial}{\partial x}\left|\partial \boldsymbol{Zx}\right\rangle \equiv \left|\partial \boldsymbol{Zxx}\right\rangle; \quad \frac{\partial}{\partial y}\left|\partial \boldsymbol{Zx}\right\rangle \equiv \left|\partial \boldsymbol{Zxy}\right\rangle; \quad \frac{\partial}{\partial y}\left|\partial \boldsymbol{Zy}\right\rangle \equiv \left|\partial \boldsymbol{Zyy}\right\rangle$$

The expressions for the three second partial derivatives of the special orthogonal functions can be written

$$\left|\partial \boldsymbol{Zxx}\right\rangle = \left[\boldsymbol{Dx}\right]\left[\boldsymbol{Dx}\right]\left|\boldsymbol{Z}\right\rangle = \left[\boldsymbol{Dx}\right]^{\boldsymbol{2}}\left|\boldsymbol{Z}\right\rangle$$

$$\left|\partial \boldsymbol{Zxy}\right\rangle = \left[\boldsymbol{Dx}\right]\left[\boldsymbol{Dy}\right]\left|\boldsymbol{Z}\right\rangle \tag{B.13}$$

$$\left|\partial \boldsymbol{Zyy}\right\rangle = \left[\boldsymbol{Dy}\right]\left[\boldsymbol{Dy}\right]\left|\boldsymbol{Z}\right\rangle = \left[\boldsymbol{Dy}\right]^{\boldsymbol{2}}\left|\boldsymbol{Z}\right\rangle$$

Note that the order of application of the transformation matrices in the mixed partial set is immaterial since for an analytic function $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$. This is also true of higher order mixed partial derivatives.

The important thing to notice in Equations (B.13) is that not only may second partial derivatives be found without first having to find any partial derivatives of the members of the special orthogonal set but also that no new coefficient matrices need be created. Having once found the elements of the first partial derivative matrices, matrix multiplication is the only additional step needed to calculate the second partial derivatives.

This formalism may be repeated for the higher order partial derivatives and a general expression for the $n$th order partial derivatives of a set of special orthogonal functions may be given as

$$\left|\partial \boldsymbol{Zx_1} \cdots \boldsymbol{x_m y_1} \cdots \boldsymbol{y_n}\right\rangle = \left[\boldsymbol{Dx}\right]^{\boldsymbol{m}}\left[\boldsymbol{Dy}\right]^{\boldsymbol{n}}\left|\boldsymbol{Z}\right\rangle \tag{B.14}$$

All necessary elements are now in place to simplify Equation (B.7).

The inner sum of Equation (B.7) normally requires a calculation of each partial derivative of the orthogonal set but as has just been shown, this is not necessary for a special orthogonal set. Therefore by identifying the partial derivative operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ with the matrix operators [$\boldsymbol{Dx}$] and [$\boldsymbol{Dy}$] respectively, Equation (B.7) can be written as

$$S(x+dx, y+dy) = \left\langle \boldsymbol{c}\right|\sum_{k}^{\infty}\frac{(-1)^k}{k!}\left(dx\left[\boldsymbol{Dx}\right] + dy\left[\boldsymbol{Dy}\right]\right)^k\left|\boldsymbol{Z}(x,y)\right\rangle \tag{B.15}$$

Here it has been recognized that Equation (B.6) has the form of an inner product so that by defining a row vector of coefficients $c_n^m$ as

$$\left|\boldsymbol{c}\right\rangle = \begin{pmatrix} c_0^0 & c_1^{-1} & \cdots & c_n^m & \cdots \end{pmatrix}$$

Equation (B.6) becomes

$$S(x,y) = \sum_{\text{all } n \text{ and } m} c_n^m Z_n^m(x,y)) = \left\langle \boldsymbol{c} \middle| \boldsymbol{Z}(x,y) \right\rangle$$

leading to the above formalism in Equation (B.13).

The quantity $\left[ \sum_k^{\infty} \frac{(-1)^k}{k!} \left( dx\left[\boldsymbol{Dx}\right] + dy\left[\boldsymbol{Dy}\right] \right)^k \right]$ is a matrix as it consists of products of matrices and the constants $k$, $dx$ and $dy$. It is not a function of position variables $x$ and $y$ so it may be evaluated separately in the sense of making it into a single matrix. This single matrix will be called the translation matrix [$\boldsymbol{T}$]. This allows Equation (B.15) to be written as

$$S(x+dx, y+dy) = \left\langle \boldsymbol{c} \middle| \left[\boldsymbol{T}\right] \middle| \boldsymbol{Z}(x,y) \right\rangle \tag{B.16}$$

where

$$\left[\boldsymbol{T}\right] = \sum_k^{\infty} \frac{(-1)^k}{k!} \left( dx\left[\boldsymbol{Dx}\right] + dy\left[\boldsymbol{Dy}\right] \right)^k \tag{B.17}$$

Since the special orthogonal sets are complete sets they have an infinite number of members but in practical use, only a limited subset is ever used. In the case of Zernike polynomials, the highest exponential power of any polynomial is the radial order of that polynomial. Therefore all derivatives of power higher than the order of a polynomial are identically zero and the value of $k$ in the $T$ sum need never be higher than the highest order found in the set chosen.

The matrix operators [$\boldsymbol{Dx}$] and [$\boldsymbol{Dy}$] for Zernike polynomial functions are quite simple. They are sparse lower triangle matrices with zeros in all elements of their diagonals. “Sparse” means that many of the elements in the lower triangular portion of the matrices are zero. Rules for finding their non-zero values are given in the paper by Noll[3] for the case of normalized Zernike functions. Normalized Zernike functions are the products of a radial polynomial, a sinusoidal meridional function and a normalization constant. The normalization constant typically takes the form of a square root and this makes many of the non-zero elements in the matrices contain square roots. However if un-normalized Zernike functions (which lack the normalization constant in the product) are used, all non-zero elements in the matrices are simple integers and so it is easier to construct matrices for the un-normalized Zernike functions. Since Zernike functions are products, when one takes the inner product of the Zernike functions and their respective coefficients to find $S(x,y)$, the normalization factors can either be included in the Zernike functions (the normalized case) or in the coefficients (the un-normalized case). Therefore if one is given coefficients in normalized form and one wishes to use the simple forms of [$\boldsymbol{Dx}$] and [$\boldsymbol{Dy}$] the normalized coefficients are converted to un-normalized coefficients by multiplying each by the correct normalization factor before use in Equation (B.13). When the Zernike functions are labelled using standard double index notation, $Z_n^m$, the normalization factor is given by

$$N_n^m = \sqrt{\left(2 - \delta_{m0}\right)\left(n+1\right)}$$

where $\delta_{m0}$ is the Kronecker delta which equals zero unless $m = 0$.

Since the creation of weighting matrices is a somewhat complex algorithmic task, a computer routine to accomplish this task is given in Annex D.

NOTE Also see Bibliography reference [6] for a method and computer routine to combine the Zernike coefficient conversions given in B.2, B.3 and B.4.

## B.4 Conversion of Zernike coefficients to account for coordinate system rotation

If the same wavefront data is analysed using Zernike decomposition, first with one coordinate system selection and then with a second coordinate system rotated with respect to the first, the coefficients obtained will be different. Therefore care must be taken, when comparing sets of Zernike coefficients or using them for some purpose with a predefined coordinate origin, that the same coordinate system is used throughout. If the amount of coordinate system rotation from the preferred coordinate system is known, the Zernike coefficients may be transformed to the preferred system using the following method.

An examination of the Zernike functions, expressed in polar coordinate form, shows that for any chosen radial index value, $n$, there are two terms for each allowed meridional index $|m|$. One index takes the value $-m$ and the other the value $+m$. These two terms may be thought of as components of a vector in a multi-angle subspace of the total Zernike functional space (see 5.1.9). The multiplicity of that subspace is the value of $|m|$. When considering the rotation of Zernike functions, it is best not to think of the functions by themselves but as pairs with the same $n$ and $|m|$ index values forming the components of a vector. It is these Zernike subspace vectors that are the items of interest.

When the Zernike functions are considered in terms of the above defined subspace vectors we see that the effect of a rotation of space is a rotation of these vectors. Their magnitudes do not change under the transformation. Only the ratio of the components of the subspace vectors change. Usually the Zernike functions are grouped in orders using their radial indices. However, to account for the effect of rotation it is better to think of them grouped into meridional subspaces by their meridional indices. Then under a rotation of physical space by angle $\phi$, the subspace vectors rotate by angle $\beta_m = |m|\phi$. Their transformed components under the transformation, the transformed Zernike coefficients, are then found by the standard formula for rotation in a two-dimensional space.

$$c_n^{m'} = \cos\beta_m c_n^m + \sin\beta_m c_n^{-m} \tag{B.18}$$

$$c_n^{-m'} = -\sin\beta_m c_n^m + \cos\beta_m c_n^{-m} \tag{B.19}$$

In matrix form this is written

$$\begin{pmatrix} c_n^{m'} \\ c_n^{-m'} \end{pmatrix} = \begin{bmatrix} \cos\beta_m & \sin\beta_m \\ -\sin\beta_m & \cos\beta_m \end{bmatrix} \begin{pmatrix} c_n^m \\ c_n^{-m} \end{pmatrix} \tag{B.20}$$

To treat the rotation of a set of Zernike coefficients, for example from the 0th to the $n$th radial order, it is most convenient to arrange the un-rotated coefficients as a column vector and operate on this vector with a special rotation matrix, $[\boldsymbol{R}]$, to yield a column vector of the rotated coefficients. The rotation matrix may be easily composed as follows. The rotation matrix is square with a number of rows and columns equal to the number of coefficients chosen. Each column of the rotation matrix is associated with an un-rotated coefficient. Each row of the rotation matrix is associated with a rotated coefficient. The elements of each row are all zero except for those elements in the two columns associated with un-rotated coefficients that are used to form the rotated coefficient of that row. If the coefficients are ordered as in the example above, the non-zero elements are those in that matrix and are arranged in that order.

Matrix [$\boldsymbol{R}$] is known as a block diagonal matrix because elements are zero except for those in a block centred on the diagonal of the matrix. In matrix [$\boldsymbol{R}$] the blocks are square. There is one block for each radial order $n$ and the size of the block is ($n$+1) square. The elements in the corners of the blocks have a value of |$m$| equal to the value of $n$ for the block. Block ($n$+2) is formed from block $n$ by adding the terms in the corners of the larger block. The centre is unchanged and other than the corners, the new elements are all zero. Cosine terms are found in the upper left and lower right quadrants. Sine terms are found in the lower left and upper right quadrants. Sine terms in the lower left quadrant are negative.

When the Zernike coefficients are given in the order specified in 6.2.3, the above described rotation matrix is represented by

$$\left[\boldsymbol{R}\right]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cos\phi & -\sin\phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sin\phi & \cos\phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cos 2\phi & 0 & -\sin 2\phi & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sin 2\phi & 0 & \cos 2\phi & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos 3\phi & 0 & 0 & -\sin 3\phi & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\phi & -\sin\phi & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\phi & \cos\phi & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin 3\phi & 0 & 0 & \cos 3\phi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{B.21}$$

The entire set of ordered Zernike coefficients is now represented by a vector, symbolised by

$$\left|\boldsymbol{c}\right\rangle=\begin{pmatrix} c_0^0 \\ c_1^{-1} \\ \vdots \\ c_n^m \\ \vdots \end{pmatrix} \tag{B.22}$$

so that the transformed coefficients, represented by a similar vector, are given by

$$\left|\boldsymbol{c}'\right\rangle=\left[\boldsymbol{R}\right]\left|\boldsymbol{c}\right\rangle \tag{B.23}$$

# Annex C
(informative)

# Conversion between Zernike coefficients represented in different systems of notation

## C.1 General

Although this International Standard defines a coordinate system for reporting aberrations of the eye, a notation for Zernike polynomials used to report these aberrations and an ordering of a set of Zernike polynomial coefficients when they are used to report these aberrations, in the field of optics several other ways to define Zernike polynomial functions exist and there is a need to connect the definitions of this International Standard to these other definitions. To this end, the following is provided to facilitate conversion from different systems to the one defined by this International Standard that will be referred to as the ISO 24157 system.

There are several different sets of definitions of Zernike polynomials that have been used over a period of fifty years. These differ primarily in three ways: normalization, coordinate system and ordering. These include the definitions given in Table C.1.

**Table C.1**

| System | Normalization [a] | Polar reference coordinate [b] | Ordering [c] |
|---|---|---|---|
| ISO 24157 | included | $x$-axis | vertical/horizontal |
| OSA ophthalmic | included | $x$-axis | vertical/horizontal |
| Zernike | separate | $y$-axis | horizontal/vertical |
| Malacara/Born and Wolf | separate | $y$-axis | horizontal/vertical |
| Noll/Arizona | | $y$-axis | special |
| Fringe | | | special |

NOTE See the Bibliography for references on the Noll[3], Malacara[2] and Born and Wolf[1] systems.

[a] Normalization refers to whether the normalization term (see 3.2.5) is included in the definition of the polynomial (and hence in the calculation of the coefficient), or is calculated separately if needed. This normalization term is adjusted so that $\sigma^2 = 1$ for each polynomial.

[b] The coordinate system refers to the reference axis for measuring the polar angle. In the Malacara/Born and Wolf systems, as well as in Zernike's original papers, this is measured from the positive $y$-axis in the clockwise direction. This is different from the normal right-handed coordinate system that is commonly used in mathematics. This International Standard uses the standardized ophthalmic coordinate system (ISO 8429) as does the OSA ophthalmic system, a right-handed coordinate system in which the angle is measured from the $x$-axis in the anticlockwise direction.

[c] The ordering refers to the order in which the various terms are listed in the table of polynomials. Because the coordinate convention of this standard differs from the Malacara/Born and Wolf coordinate convention while maintaining the same Zernike definitions in polar form, the ordering of the odd terms in this International Standard is reversed from the Malacara/Born and Wolf set. Thus the vertical coma appears first in the table instead of the horizontal coma. The ordering may thus be distinguished by the order of the coma terms. This change in ordering also results in a change in sign of some of the terms. In the case of the Noll/Arizona set, the order has been specifically chosen so that the symmetric terms appear earlier in the series to match the order that these effects are commonly seen in optical systems.

## C.2 Conversion between a Malacara/Born and Wolf coefficient set and an ISO 24157 coefficient set

The polynomials that are described in one set may readily be converted to another set using the following formulae.

**Conversion from ISO 24157 $\left(Z_n^m\right)$ to Malacara/Born and Wolf $\left(V_n^l\right)$**

$$l = (-1)^n m$$

$$\text{scale} = \begin{cases} \dfrac{m}{|m|}(-1)^{|m|/2} & \text{for } m \text{ even} \\ (-1)^{(|m|-1)/2} & \text{for } m \text{ odd} \end{cases}$$

$$V_n^l = (\text{scale})\sqrt{(2-\delta_{m0})(n+1)}Z_n^m \qquad \text{(C.1)}$$

where $\delta_{m0} = 1$ if $m = 0$, otherwise $\delta_{m0} = 0$

**Conversion from Malacara/Born and Wolf $\left(V_n^l\right)$ to ISO 24157 $\left(Z_n^m\right)$**

$$m = (-1)^n l$$

$$\text{scale} = \begin{cases} \dfrac{l}{|l|}(-1)^{|l|/2} & \text{for } l \text{ even} \\ (-1)^{(|l|-1)/2} & \text{for } l \text{ odd} \end{cases}$$

$$Z_n^m = \frac{(\text{scale})V_n^l}{\sqrt{(2-\delta_{l0})(n+1)}} \qquad \text{(C.2)}$$

where $\delta_{l0} = 1$ if $l = 0$, otherwise $\delta_{l0} = 0$

NOTE Malacara also uses the notation ($U_{nm}$) where $m = (n-l)/2$. This "$m$" is therefore different from the ISO 24157 "$m$". Born and Wolf also use $m = |l|$ so this "$m$" is the absolute value of the ISO 24157 "$m$".

# Annex D
(informative)

# Computer algorithm to generate partial derivative weighting matrices for un-normalized Zernike polynomial functions

The following is an example of a routine, written in MatLab code, to generate the partial weighting matrices needed to implement the method given in B.3 for the conversion of Zernike coefficients to account for decentration of the coordinate origin.

```
function [Dx,Dy]=zpartials(order)
%ZPARTIALS creates x and y partial derivative weighting
%   matrices, Dx and Dy, which allow the partial derivatives
%   of un-normalized Zernike polynomial functions to be formed,
%   through the specified radial order (order),
%   as sums of the Zernike functions themselves.
%   [Dx,Dy]=zpartials(order);


terms=fix(.5*order*(order+3))+1;      % number of terms
Dx=zeros(terms);    %   initialize the x partial derivative matrix
Dy=zeros(terms);    %   initialize the y partial derivative matrix
r=0;      %   intialize the row index
c=0;      %   intialize the column index


for i=1:order+1   % i= n+1 where n is the radial index of the function
  % whose partial derivative is to be formed
  for j=1:i  %      index associated with the azimuthal index m
    r=r+1;
    m=2*j-i-1;
    if m==0
      delta=1;
    else
      delta=0;
    end
    mm=abs(m)-1;
    mp=abs(m)+1;
    s=sign(m);
    if s==0
```

```
    s=1;
  end

  if mm>0
    for np=mm:2:i-2
      cx=fix(.5*(np*(np+2)+s*mm))+1;
      cy=fix(.5*(np*(np+2)-s*mm))+1;
      Dx(r,cx)=(1+delta)*(np+1);
      Dy(r,cy)=-s*(1+delta)*(np+1);
    end
  end
  if mm==0&s>0
    for np=mm:2:i-2
      cx=fix(.5*np*(np+2))+1;
      Dx(r,cx)=(1+delta)*(np+1);
    end
  end
  if mm==0&s<0
    for np=mm:2:i-2
      cy=fix(.5*np*(np+2))+1;
      Dy(r,cy)=-s*(1+delta)*(np+1);
    end
  end
  for np=mp:2:i-2
    cx=fix(.5*(np*(np+2)+s*mp))+1;
    cy=fix(.5*(np*(np+2)-s*mp))+1;
    Dx(r,cx)=(1+delta)*(np+1);
    Dy(r,cy)=s*(1+delta)*(np+1);
  end
 end
end
```

# Annex E
(informative)

# Table of normalized Zernike polynomial functions (to 6th radial order)

**Table E.1**

| Symbol | Polar form | Cartesian form | Common name |
|---|---|---|---|
| $Z_0^0$ | 1 | 1 | piston – mean elevation |
| $Z_1^{-1}$ | $2\rho\sin(\theta)$ | $2y$ | vertical tilt |
| $Z_1^1$ | $2\rho\cos(\theta)$ | $2x$ | horizontal tilt |
| $Z_2^{-2}$ | $\sqrt{6}\rho^2\sin(2\theta)$ | $2\sqrt{6}xy$ | oblique astigmatism |
| $Z_2^0$ | $\sqrt{3}(2\rho^2-1)$ | $\sqrt{3}(2x^2+2y^2-1)$ | myopic defocus (positive coefficient value)<br>hyperopic defocus (negative coefficient value) |
| $Z_2^2$ | $\sqrt{6}\rho^2\cos(2\theta)$ | $\sqrt{6}(x^2-y^2)$ | against the rule astigmatism (positive coefficient value)<br>with the rule astigmatism (negative coefficient value) |
| $Z_3^{-3}$ | $\sqrt{8}\rho^3\sin(3\theta)$ | $\sqrt{8}(3x^2y-y^3)$ | oblique trefoil |
| $Z_3^{-1}$ | $\sqrt{8}(3\rho^3-2\rho)\sin(\theta)$ | $\sqrt{8}(3x^2y+3y^3-2y)$ | vertical coma – superior steepening (positive coefficient value)<br>vertical coma – inferior steepening (negative coefficient value) |
| $Z_3^1$ | $\sqrt{8}(3\rho^3-2\rho)\cos(\theta)$ | $\sqrt{8}(3x^3+3xy^2-2x)$ | horizontal coma |
| $Z_3^3$ | $\sqrt{8}\rho^3\cos(3\theta)$ | $\sqrt{8}(x^3-3xy^2)$ | horizontal trefoil |
| $Z_4^{-4}$ | $\sqrt{10}\rho^4\sin(4\theta)$ | $\sqrt{10}(4x^3y-4xy^3)$ | oblique quatrefoil |
| $Z_4^{-2}$ | $\sqrt{10}(4\rho^4-3\rho^2)\sin(2\theta)$ | $\sqrt{10}(8x^3y+8xy^3-6xy)$ | oblique secondary astigmatism |
| $Z_4^0$ | $\sqrt{5}(6\rho^4-6\rho^2+1)$ | $\sqrt{5}(6x^4+12x^2y^2+6y^4-6x^2-6y^2+1)$ | spherical aberration<br>positive coefficient value – pupil periphery more myopic than centre<br>negative coefficient value – pupil periphery more hyperopic than centre |
| $Z_4^2$ | $\sqrt{10}(4\rho^4-3\rho^2)\cos(2\theta)$ | $\sqrt{10}(4x^4-4y^4-3x^2+3y^2)$ | with/against the rule secondary astigmatism |

**Table E.1** (*continued*)

| Symbol | Polar form | Cartesian form | Common name |
|---|---|---|---|
| $Z_4^4$ | $\sqrt{10}\rho^4\cos(4\theta)$ | $\sqrt{10}(x^4-6x^2y^2+y^4)$ | horizontal quatrefoil |
| $Z_5^{-5}$ | $\sqrt{12}\rho^5\sin(5\theta)$ | $\sqrt{12}(5x^4y-10x^2y^3+y^5)$ | |
| $Z_5^{-3}$ | $\sqrt{12}(5\rho^5-4\rho^5)\sin(3\theta)$ | $\sqrt{12}(15x^4y+10x^2y^3-12x^2y-5y^5+4y^3)$ | |
| $Z_5^{-1}$ | $\sqrt{12}(10\rho^5-124\rho^3+3\rho)\sin(\theta)$ | $\sqrt{12}(10x^4y+20x^2y^3+10y^5-12x^2y-12y^3+3y)$ | |
| $Z_5^{1}$ | $\sqrt{12}(10\rho^5-124\rho^3+3\rho)\cos(\theta)$ | $\sqrt{12}(10x^5+20x^3y^2+10xy^4-12xy^2-12x^3+3x)$ | |
| $Z_5^{3}$ | $\sqrt{12}(5\rho^5-4\rho^5)\cos(3\theta)$ | $\sqrt{12}(5x^5-10x^3y^2+12xy^2-15xy^4-4x^3)$ | |
| $Z_5^{5}$ | $\sqrt{12}\rho^5\cos(5\theta)$ | $\sqrt{12}(x^5-10x^3y^2+5xy^4)$ | |
| $Z_6^{-6}$ | $\sqrt{14}\rho^6\sin(6\theta)$ | $\sqrt{14}(6x^5y-20x^3y^3+6xy^5)$ | |
| $Z_6^{-4}$ | $\sqrt{14}(6\rho^6-5\rho^4)\sin(4\theta)$ | $\sqrt{14}(24x^5y-20x^3y-24xy^5+20xy^3)$ | |
| $Z_6^{-2}$ | $\sqrt{14}(15\rho^6-20\rho^4+6\rho^2)\sin(2\theta)$ | $\sqrt{14}(30x^5y+60x^3y^3+30xy^5-40x^3y-40xy^3+12xy)$ | |
| $Z_6^{0}$ | $\sqrt{7}(20\rho^6-30\rho^4+12\rho^2-1)$ | $\sqrt{7}(20x^6+60x^4y^2+60x^2y^4+20y^6-30x^4-60x^2y^2$ $-30y^4+12x^2+12y^2-1)$ | |
| $Z_6^{2}$ | $\sqrt{14}(15\rho^6-20\rho^4+6\rho^2)\cos(2\theta)$ | $\sqrt{14}(15x^6+15x^4y^2-15x^2y^4-15y^6-20x^4+20y^4+6x^2$ $-6y^2)$ | |
| $Z_6^{4}$ | $\sqrt{14}(6\rho^6-5\rho^4)\cos(4\theta)$ | $\sqrt{14}(6x^6-30x^4y^2-30x^2y^4+6y^6-5x^4+30x^2y^2-5y^4)$ | |
| $Z_6^{6}$ | $\sqrt{14}\rho^6\cos(6\theta)$ | $\sqrt{14}(x^6-15x^4y^2+15x^2y^4-y^6)$ | |

NOTE 1 The variable $\rho$ may only take values between 0 and 1. The variables $x$ and $y$ are limited to values such that $(x^2+y^2)$ may only take values between 0 and 1.

NOTE 2 The Zernike coefficients represent the wavefront error, whereas normally in ophthalmic optics the correction to that error is considered. Thus the axis of astigmatism, expressed in the above table in which the Zernike coefficients represent the wavefront error instead of the correction to that error, is rotated by 90° from the common ophthalmic convention, when the astigmatism is expressed in minus cylinder form.

# Bibliography

[1] BORN, M. and WOLF, E., *Principles of Optics*, Pergamon Press, Oxford, Sec. 9.2 and Appendix VII, 1980

[2] MALACARA, D., *Optical Shop Testing*, Wiley, New York, Sec. 13.2.2 and Appendix 2, 1978

[3] NOLL, R.J., Zernike polynomials and atmospheric turbulence, *J. Opt. Soc. Am.*, **66**, pp. 207-211, 1976

[4] ZERNIKE, F., *Physica* **1**, p. 689, 1934

[5] ZERNIKE, F. and BRINKMAN, H.C., Verh. Akad. Wet. Amst., *Proc. Sec. Sci.*, **38**, p. 11, 1935

[6] LUNDSTRÖM, L. and UNSBO, P., *Transformation of Zernike coefficients: scaled, translated, and rotated wavefronts with circular and elliptical pupils*, J. Opt. Soc. Am. A, **24**, pp. 569-577, 2007

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