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Statistical methods in process management — Capability and performance —

Part 6: Process capability statistics for characteristics following a multivariate normal distribution

Méthodes statistiques dans la gestion des processus — Capabilité et performance —

Partie 6: Statistiques de capabilité pour un processus caractérisé par une distribution normale multivariée

Reference number
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Contents

Page

Foreword

ISO (the International Organization for Standardization) is a worldwide federation of national standards bodies (ISO member bodies). The work of preparing International Standards is normally carried out through ISO technical committees. Each member body interested in a subject for which a technical committee has been established has the right to be represented on that committee. International organizations, governmental and non-governmental, in liaison with ISO, also take part in the work. ISO collaborates closely with the International Electrotechnical Commission (IEC) on all matters of electrotechnical standardization.

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The main task of technical committees is to prepare International Standards. Draft International Standards adopted by the technical committees are circulated to the member bodies for voting. Publication as an International Standard requires approval by at least 75 % of the member bodies casting a vote.

Attention is drawn to the possibility that some of the elements of this document may be the subject of patent rights. ISO shall not be held responsible for identifying any or all such patent rights.

ISO 22514-6 was prepared by Technical Committee ISO/TC 69, *Applications of statistical methods*, Subcommittee SC 4, *Applications of statistical methods in process management*.

ISO 22514 consists of the following parts, under the general title *Statistical methods in process management — Capability and performance*:

— *Part 1: General principles and concepts*

— *Part 2: Process capability and performance of time-dependent process models*

— *Part 3: Machine performance studies for measured data on discrete parts*

— *Part 4: Process capability estimates and performance measures* [Technical Report]

— *Part 5: Process capability statistics for attribute characteristics*

— *Part 6: Process capability statistics for characteristics following a multivariate normal distribution*

— *Part 7: Capability of measurement processes*

— *Part 8: Machine performance of a multi-state production process*

Introduction

Due to the increased complexity of the production methods and the increasing quality requirements for products and processes, a process analysis based on univariate quantities is in many cases not sufficient.

Instead, it may be necessary to analyse the process on the basis of multivariate product quantities. This can, for instance, be in such cases where geometric tolerances, dynamic magnitudes such as imbalance, correlated quantities of materials or other procedural products are observed.

By analogy with ISO 22514-2, ISO 22514-6 provides calculation formulae for process performance and process capability indices, which take into consideration process dispersion as well as process location as an extension to the corresponding indices for univariate quantities. The indices proposed are indeed based on the classical C_{p} and C_{pk} indices for the one-dimensional case. The motivation for the extension to the multivariate case is explained in Annex A.

Examples of possible applications are two-dimensional or three-dimensional positions, imbalance or several correlated quantities of chemical products.

The dispersion of the measuring results comprises the dispersion of the product realization process and the precision of the measuring process. It is assumed that the capability of the used measuring system was demonstrated prior to the determination of the capability of the product realization process.

The calculation method described here should be used to support an unambiguous decision, especially if

— limiting values for process capability indices for multivariate, continuous product quantities are specified as part of a contract between customers and suppliers, or

— the capabilities of different constructions, production methods or suppliers are to be compared, or

— production processes are to be approved, or

— problems are to be analysed and decisions made in complaint cases or damage events.

NOTE Product realization processes include e.g. manufacturing processes, service processes, product assembly processes.

Statistical methods in process management — Capability and performance —

Part 6:
Process capability statistics for characteristics following a multivariate normal distribution

1 Scope

This part of ISO 22514 provides methods for calculating performance and capability statistics for process or product quantities where it is necessary or beneficial to consider a family of singular quantities in relation to each other. The methods provided here mostly are designed to describe quantities that follow a bivariate normal distribution.

NOTE In principle, this part of ISO 22514 can be used for multivariate cases.

This part of ISO 22514 does not offer an evaluation of the different provided methods with respect to different situations of possible application of each method. For the current state, the selection of one preferable method might be done following the users preferences.

The purpose is to give definitions for different approaches of index calculation for performance and capability in the case of a multiple process or product quantity description.

2 Normative references

The following documents, in whole or in part, are normatively referenced in this document and are indispensable for its application. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

ISO 22514-1, *Statistical methods in process management — Capability and performance — Part 1: General principles and concepts*

ISO 22514-2, *Statistical methods in process management — Capability and performance — Part 2: Process capability and performance of time-dependent process models*

3 Terms and definitions

For the purpose of this document, the terms and definitions given in ISO 22514-1 and ISO 22514-2 and the following apply.

3.1
quantity
property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference

[ISO/IEC Guide 99:2007, 1.1]

3.2
multivariate quantity
set of distinguishing features

Note 1 to entry: The set can be expressed by a d-tuple, i.e. an ordered set consisting of d elements.

Note 2 to entry: If the single quantities in the set are denoted by x_i where $i = 1, 2\ldots d$, the multivariate quantity is expressed as the vector $\mathbf{x} = (x_1, x_2, \ldots x_d)^T$. Thus, a multivariate quantity can be considered as a feature vector of a product. The value of the multivariate quantity is represented by a point in the *d*-dimensional feature space.

Note 3 to entry: The selection of the quantities in a vector is made for specific technical reason.

Note 4 to entry: All single quantities combined in the vector of a multivariate must be measurable in the same product or object.

Note 5 to entry: If the multivariate quantity is to be described by means of statistics, the vector is to be considered as a random vector following a *d*-dimensional multivariate distribution.

EXAMPLE 1 A number of $d = 3$ quantities like x_1 = colour, x_2 = mass and x_3 = number of defects are combined in order to use only one statistic for process assessment. The dimension of vector **x** is $d = 3$.

EXAMPLE 2 In order to evaluate a boring process, the position of the borehole axis is measured in an *x*-coordinate and *y*-coordinate. The coordinates are combined to the two-dimensional multivariate quantity **x** where the component x_1 is the *x*-coordinate and x_2 is the *y*-coordinate.

EXAMPLE 3 Imbalance of a wheel.

3.3
tolerance region
region in the feature space that contains all permitted values of the *multivariate quantity* (3.2)

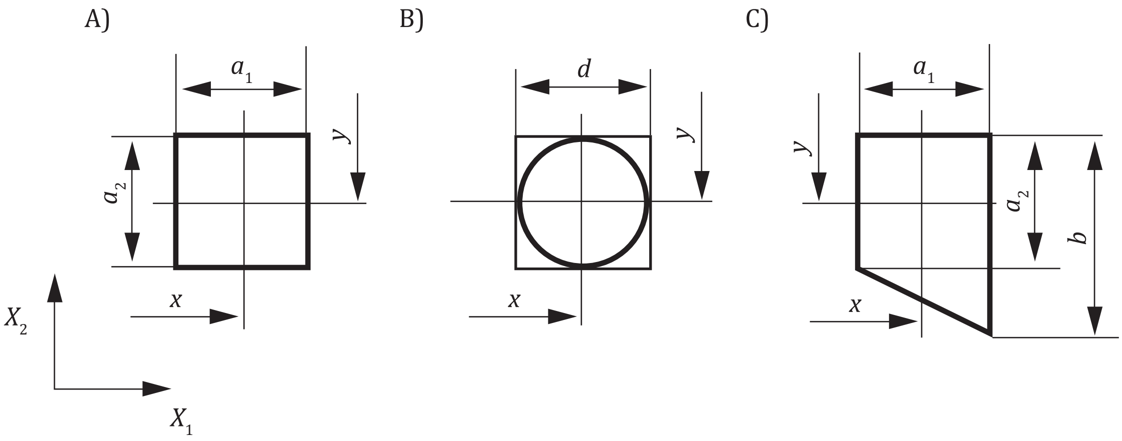
Note 1 to entry: The region is limited by lines, surfaces or hyper-surfaces in the *d*-dimensional space and not necessarily closed. The form and extension of the region are specified by one or more parameters.

Note 2 to entry: Typical shapes of tolerance regions are rectangles, ellipses (or circles) in the two-dimensional case, cuboids or hyper-cuboids, ellipsoids or hyper-ellipsoids or composite prismatic shapes. Figure 1 shows examples of tolerance regions in the two-dimensional space.

Note 3 to entry: The tolerance region is specified based on the required function of the product. Products showing values outside the region are assumed to not fulfil functional requirements. Those products are considered to be nonconforming parts.

Note 4 to entry: In order to assess a product with respect to the limits of the tolerance region, the order of the single quantity in the multivariate quantity and the number *d* of dimension must be equal to that of the tolerance region description.

EXAMPLE A tolerance zone as it is defined in ISO 1101 for geometrical product features can be considered as a tolerance region. In that case, limiting geometrically perfect lines or surfaces correspond to the boundary and the tolerance correspond to the parameter of the tolerance region.



Key

A rectangular tolerance region with parameters a_1, a_2, x and y

B circular tolerance region with parameters d, x and y

C triangularly extended rectangular region with parameters a_1, a_2, b, x and y

Figure 1 — Examples of tolerance regions in the two-dimensional space of the bivariate quantity $(x_1, x_2)^T$

3.4
process capability
distribution of measured *quantity* (3.1) values from a process that has been demonstrated to be in statistical control and which describes the ability of a process to produce quantity values that will fulfil the requirements for that quantity

Note 1 to entry: The process capability index provides the ability to meet requirements of the measured quantity.

Note 2 to entry: The abbreviation for process capability index is PCI.

3.5
estimated process capability
statistical description of a *process capability* (3.4)

3.6
process performance
distribution of measured *quantity* (3.1) values from a process

Note 1 to entry: The process may not have been demonstrated to be in statistical control.

3.7
estimated process performance
statistical description of a *process performance* (3.6)

4 Abbreviated terms

MMC maximum material condition

PCI process capability index

5 Process analysis

The purpose of process analysis is to obtain sound knowledge of a process. This knowledge is necessary for controlling the process efficiently, so that the products realized by the process fulfil the quality requirement.

A process analysis is always an analysis of one or more quantities that are considered to be important to the process.

Product quantities can often be analysed instead of process quantities because product quantities not only characterize the products, but due to their correlation with process quantities they also characterize the process creating these products.

The values of the quantities under consideration are typically determined on the basis of samples taken from the process flow. The sample size and frequency should be chosen depending on the type of process and the type of product so that all important changes are detected in time. The samples should be representative for the multivariate quantities under consideration. (Univariate quantity values are considered in ISO 22514-2.) This part of ISO 22514 describes multivariate capability statistics.

To estimate the PCI, the sample size should preferably be at least 125.

6 Use of multivariate process capability and performance assessment

The purpose of a process capability index is to reflect how well or how badly a process generates qualified products. The use of PCI for multivariate quantities should reflect this process behaviour better than PCI for single quantities would. Since a variety of multivariate PCI definitions exists, the selection of a specific definition to be used will remain in the user's accountability. However, the following guidance is given as to when a multivariate PCI should be preferred at all.

A multivariate assessment of process capability and performance is suitable if at least one of the following circumstances is applicable.

— It is found to be advantageous to describe process capability and performance with only one comprehensive statistic instead of a high number of single statistics for each product quality quantity.

— The boundary of the tolerance region cannot be expressed independently for all quantities, i.e. at least one tolerance limit for one quantity is a function of another quantity. This is the case if the tolerance region is not of rectangular or cuboid shape.

— The single quantities that could be combined to a multivariate one appear to be correlated among each other.

EXAMPLE In the case of a two-dimensional position tolerance for a borehole axis, the tolerance region is a circle with defined distances in an x- and y-coordinate direction from the references, see 8.1. The result of the hole axis measurement will be a value for the x- and y-coordinate. The tolerance limit for the x-coordinate cannot be expressed independently from the y-coordinate. Thus, a bivariate assessment is to be applied.

7 Calculation of process capability and process performance

7.1 Description of Types I and II

In the multivariate domain, different approaches exist for measuring process capability and process performance. This part of ISO 22514 describes examples of two different types of indices: Type I and Type II. The distinction between the types is based on whether the index is defined based on probability or defined geometrically by relating the area or volume of a tolerance or process region.

The following description of the types applies:

— Type I Based on the probability of conforming or non-conforming products P, the index is calculated using the relationship between the index and the said probability in a univariate normal case.

— Type II The index is calculated as the ratio of the area or volume of the tolerance region to the area or volume of the region covered by the process variation.

For practical reasons, the multivariate normal distribution mode has been chosen for the calculation of the statistics which are described in this clause. However, the choice of normal distribution does not exclude that in special cases other model distributions will describe the reality better. Also, for practical reasons, in this part of ISO 22514, the process variation region has been chosen to be of ellipsoid shape.

The most important properties of the multivariate normal distribution are explained in Annex A.

Because of that choice, additional transformations should be applied to make the shape of the process variation intervals comparable to the shape of the tolerance region. Thus, three further principles are to be distinguished. These are the principles of transforming the shape of the

a) tolerance region into the shape of the process variation interval,

b) process variation interval into the shape of the tolerance region, and

c) tolerance region and/or the process variation into a new function-oriented dimension.

Both the above-mentioned types and the principles can be combined to define a multivariate PCI. Each combination, however, may not be useful. There is, for instance, no known definition of a type Ib PCI.

The term “capability” can only be used for processes that have been demonstrated to be in statistical control using control charts. In the multivariate case, the distinction between special and common causes is usually more difficult than in the univariate case. If the process has not been demonstrated to be in statistical control, the term “performance” is used in this part of ISO 22514.

7.2 Designation and symbols of the indices

7.2.1 General

Different symbols are currently used for multivariate index definitions in industry and science. Currently used symbols try to distinguish between the types of calculation or to specify their use. This part of ISO 22514 uses the designation C_p and/or C_pk for basic definitions of calculation. Furthermore, it will be distinguished between process capability and process performance in applying the indices by using capitals “C” for capability and “P” for performance.

7.2.2 Process capability index

Consider a d-dimensional normal distribution $N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. If the tolerance region is not of elliptic shape (circle, ellipse if $d = 2$ or sphere, ellipsoid if $d = 3$ or hyper-sphere, hyper-ellipsoid if $d > 3$), it is to be transformed into a modified tolerance region that is of elliptic shape. This is to be done by determining the largest ellipse (or ellipsoid, hyper-ellipsoid) that is centred at the target and completely fits into the original tolerance region.

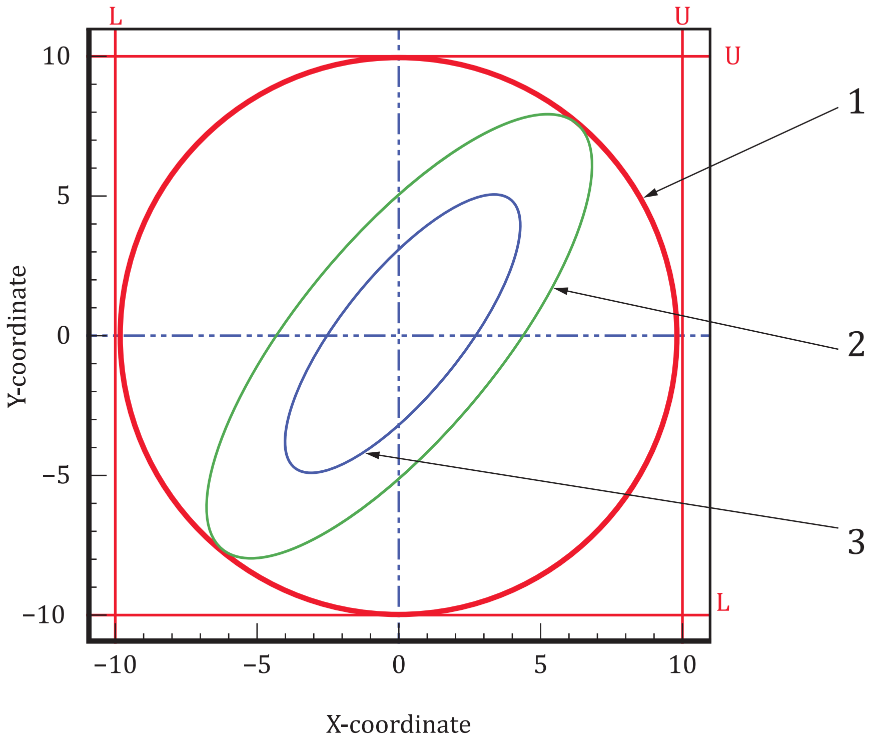
In order to calculate the multivariate C_p index, the normal distribution shall be centred to have the mean at the centre of the elliptic tolerance region. For that normal distribution, determine the largest contour ellipsoid that is completely contained in the elliptic tolerance region and calculate the probability of the volume bounded by that contour ellipse under the d-dimensional normal distribution with covariance matrix $\boldsymbol{\Sigma}$ and mean at the centre of the elliptic tolerance region. Denote that probability by P. Then, the multivariate C_p index is

$$C_\mathrm{p} = \frac{1}{3}\Phi^{-1}\left(\frac{P+1}{2}\right)$$

The calculation of P, the probability for observations of $\underline{x}$ within the determined contour ellipse (ellipsoid/hyper-ellipsoid) for any d can be done by using the relation to the F-distribution. The explanation is given in Clause A.1.

In order to estimate a C_p index from d-dimensional data, start by estimating the covariance matrix of the multivariate normal distribution from the data. Denote the estimate by $\hat{\Sigma}$ and use that covariance matrix to determine the contour ellipsoid and its probability $\hat{P}$. Finally, the estimated multivariate C_p index is

$$\hat{C}_p = \frac{1}{3}\Phi^{-1}\left(\frac{\hat{P}+1}{2}\right)$$



Key

1 elliptic tolerance zone

2 contour ellipse used for the calculation of the capability index

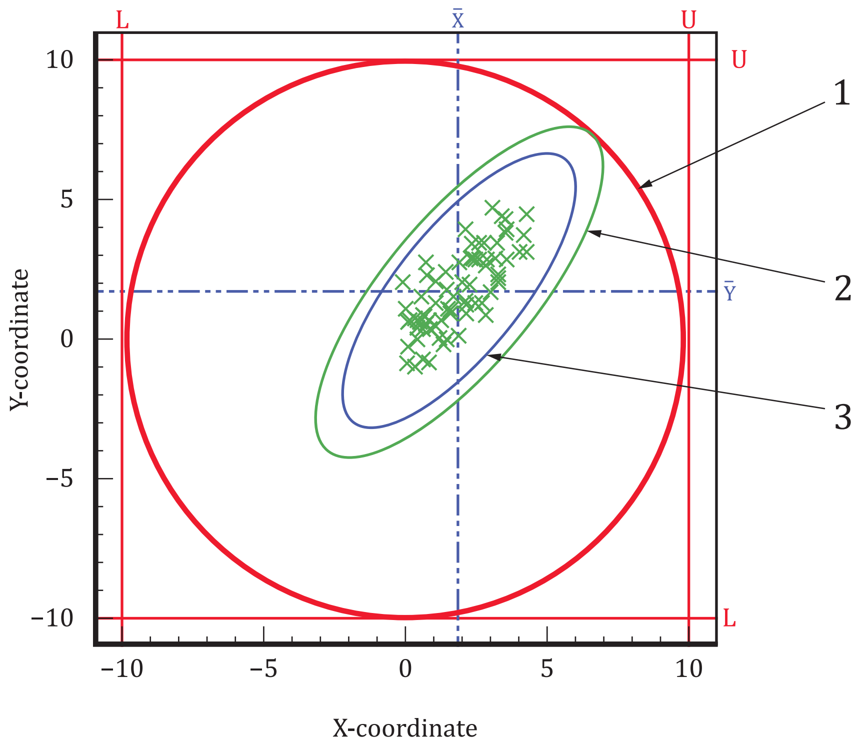
3 contour ellipse corresponding to the probability zone 99,73 %

Figure 2 — Contour ellipse and tolerance zone used to calculate the capability index for d = 2

In Figure 1, the contour ellipse with probability 99,73 % is completely contained in the contour ellipse used for the calculation of the index. When this is the case, the index will be larger than 1.

We use the symbol C_p for this index as for the classical capability index for the univariate normal distribution. The reason is that this calculation method in the one-dimensional case gives the classical C_p index. This is explained in Clause A.1.

7.2.3 Minimum process capability index



Key

1 elliptic tolerance zone

2 contour ellipse used for the calculation of the C_{pk} index

3 contour ellipse corresponding to the probability 99,73 %

Figure 3 — Tolerance zone and contour ellipse used to calculate the capability index for $d = 2$

Calculation of the C_{pk} index involves both the mean and the variance of the distribution, so consider again a d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. For the $N_d(\boldsymbol{\mu},\boldsymbol{\Sigma})$ distribution, calculate

— the largest contour ellipse (ellipsoid, hyper-ellipsoid) that is completely contained in the elliptic tolerance region, if $\boldsymbol{\mu}$ is contained in the tolerance region, or

— the largest contour ellipse (ellipsoid, hyper-ellipsoid) that is not contained in the tolerance region, if $\boldsymbol{\mu}$ is not contained in the tolerance zone.

Now, the probability, P, of the area (volume) contained in the contour ellipse (ellipsoid, hyper-ellipsoid) under the $N_d(\boldsymbol{\mu},\boldsymbol{\Sigma})$ distribution is calculated. Finally, the C_{pk} index is calculated as

$$C_{pk} = \frac{1}{3}\Phi^{-1}\left(\frac{P+1}{2}\right)$$

if $\boldsymbol{\mu}$ is in the tolerance region and as

$$C_{pk} = \frac{1}{3}\Phi^{-1}\left(\frac{1-P}{2}\right)$$

if $\boldsymbol{\mu}$ is not in the tolerance region.

We use the same symbol as for the classical C_{pk} index for the one-dimensional normal distribution. The reason is that this method of calculation gives the classical index in the one-dimensional case. This is explained in Clause A.1.

NOTE The described Type Ia finds applications in geometrical dimensioning and tolerancing of position deviations. Here, the tolerance region usually describes a circular tolerance zone. The symbols often used in that case are C_{Po} and C_{Pok} for C_p and C_{pk} respectively.

7.3 Types Ic and IIc process capability index

Type Ic as well as Type IIc capability indices are characterized by a function-oriented transformation of the multiple feature characterization into a single feature characterization. By that type, the multivariate aspect is expressed in the definition of the transforming function $q(\boldsymbol{x})$, where $\boldsymbol{x}$ describes the multivariate quantity. This transformation shall represent the functional importance of the single quantities in $\boldsymbol{x}$ and their interplay. For example, it describes a model for the tolerance region and can be interpreted as a weighing function, e.g. a loss function or a quantification function that quantifies the technical functionality.

The calculation of Type Ic and IIc indices follows four steps; see Figure 4.

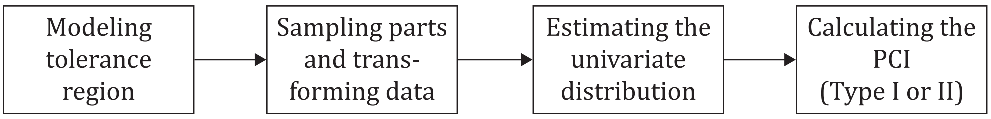
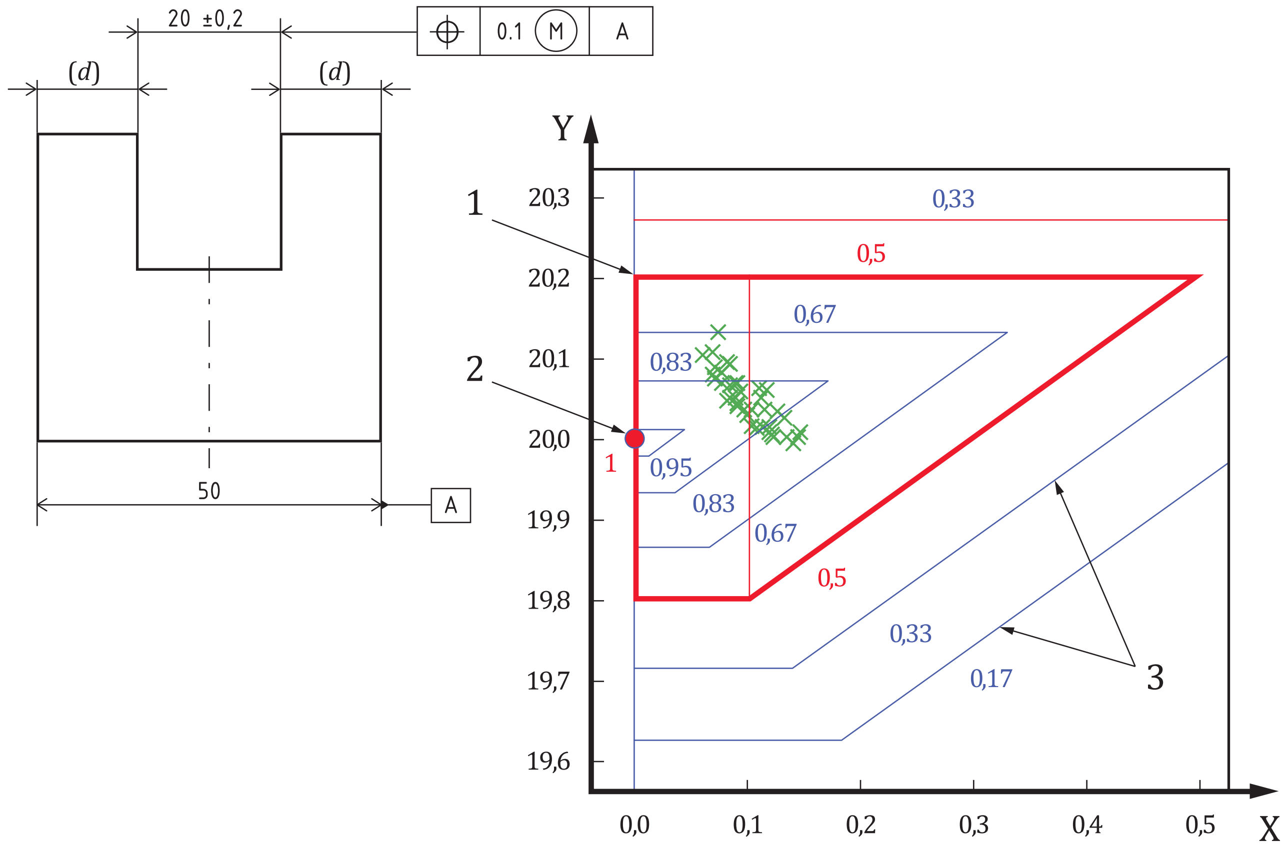


Figure 4 — Steps for calculating the type Ic or IIc process capability indices

The first step concerns the definition of the technical qualification function $q(\boldsymbol{x})$ over the d-dimensional tolerance region. This function has a maximum with the value q_{max} at the target in the tolerance region. At the boundary of the tolerance region, the $q(\boldsymbol{x})$ has the value q_{bound}. In some cases, q_{max} and q_{bound} can be derived from the technical context of all single quantities in $\boldsymbol{x}$. In other cases, appropriate values are q_{max} = 1 and q_{bound} = 0,5. The function $q(\boldsymbol{x})$ may be expressed in a closed equation or piecewise composed. An example for a piecewise composed linear function is given in Figure 5.

Dimenstions in millimetres



Key

X position

Y width

1 q_{bound} - contour; limits of the tolerance region

2 q_{max} = 1, target

3 $q(\underline{x})$ contour lines

Figure 5 — Example of a qualification function for a width/symmetry tolerance region under MMC

In Figure 5, the multivariate quantity consists of two geometrical features: width and position. The tolerance region by utilizing the maximum-material-condition is of composite rectangular and triangular shape. Further explanations about the example are given in Clause 8. The target where q_{max} = 1 is situated at the nominal values. From that point, the qualification function is decreasing towards the tolerance limits. Thereby different trends may be defined: linear, exponential and others. In Figure 5, the function $q(\boldsymbol{x})$ is composed of three linear functions. Completely different constructions of $q(\boldsymbol{x})$ are also possible.

In the second step, produced parts are to be sampled and measured. The measured values then are to be complied to the multivariate quantity $\boldsymbol{x}$ and transformed by $q(\boldsymbol{x})$ into function-related qualification values.

In the third step, based on these values, an appropriate univariate distribution function $F(q)$ is to be identified. Alternatively, a second transformation to univariate normal may be carried out. If the qualification function $q(\boldsymbol{x})$ is monotonically increasing from the boundary to the target and the random vector **X** follows a multivariate normal, the distribution density of $F(q)$ will be unimodal.

In the fourth step, based on the identified distribution of the qualification values, the transformed target and the specification limits, the PCI is calculated. If q_{max} = 1 and q_{bound} = 0,5 are chosen, 0,5 gives the lower specification limit and 1 is the upper natural limit. Since q offers only a one-sided tolerance interval and a one-sided limited distribution, one might only use C_{pk} for a process assessment.

If there is an **I-type** index, C_{pk} is calculated analogously to 7.2.3. The probability P is given from the univariate distribution $F(q)$ by $P = 1 - F(q_{bound})$.

If a Type **II** index is to be calculated, the methods in ISO 22514-2 can be followed. Based on the definition for C_{pkL}, by choosing X_{mid} as the median of $F(q)$, choosing L as q_{bound} and $\Delta L = X_{mid} - X_{0,135\,\%}$, the equation for C_{pk} is given by:

$$C_{pk} = \frac{q_{50\%} - q_{bound}}{q_{50\%} - q_{0,135\%}}$$

where $q_{x\%}$ describe the x % - percentiles of $F(q)$.

Since the C_{pk} carries information about both variation and location of the process in relation to the tolerance limit q_{bound}, one may require an index only for the information about variation. A C_p can be calculated applying the methods in ISO 22514-2. But since the target value of q is the maximum value, C_p may be lower than C_{pk}.

If, in step 1, a loss function $l(\mathbf{x})$ is defined instead of a qualification function, the function will have a minimum l_{min} at the functional target and a value l_{bound} at the limits of the tolerance region. Although it is not a loss function by the original meaning, the distances D of borehole axes from a target position can be interpreted as a $l(\mathbf{x})$ - function. It has a minimum of $l_{min} = 0$ at the target and l_{bound} at the half of the position tolerance. The tolerance region is a circle.

7.4 Types IIa and Type IIb process capability index

7.4.1 General

Multivariate process capability indices of Type II follow the principle of putting in relation an extent of the tolerance region to an extent of the process variation. These extents are expressed in areas or volumes. The area or volume of the tolerance region is denoted by V_{tol} and V_{proc} denotes the area or volume of the process variation region. Thus, for the index, the following can be defined:

$$C_p = \left[\frac{V_{tol}}{V_{proc}}\right]^a$$

The exponent a is introduced to give the possibility of reducing the area or volume back to one dimension. Thus, a is normally $a = 1/d$. Otherwise, a is set to $a = 1$.

In order to make the areas or volumes comparable, a transformation of the shape of the regions may be necessary. Type IIa indices transform the original tolerance region into a modified tolerance region that is of the shape of the process variation region (e.g. elliptic/ ellipsoid/ hyper-ellipsoid in the case of a multivariate normal distribution). For Type IIb indices, a transformation is made for the process variation region. In this case, the shape of the process variation region is adapted to the shape of the tolerance region. Comparisons between IIa and IIb are given in Reference.[9]

Since this index only gives information about the process variance in relation to tolerance, the index may be supported by one or more indices giving information about the location of the mean vector $\boldsymbol{\mu}$ in relation to the target.

7.4.2 Type IIa

The modified tolerance region is again defined as the largest ellipse (or ellipsoid, hyper-ellipsoid) that is centred at the target and completely contained within the original tolerance region; see Figure 2 and

Figure 3. In the case of a tolerance region where the parameters x_{t1}, x_{t2} ... x_{td} denote the half distance from the tolerance limit to the target, the volume V_{tol} is given by:

$$V_{\text{tol}} = \frac{\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)} \cdot \prod_{i=1}^{d} x_{ti}$$

For estimating the C_p, the volume of the 99,73 % ellipsoid is to be estimated by:

$$V_{\text{proc}} = \frac{\left(\pi \cdot \chi^2_{99,73}\right)^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)} \cdot \sqrt{|\boldsymbol{S}|}$$

$|\boldsymbol{S}|$ is the determinant of S (Annex A). This index is supported by the value of $1/D$ to give an information about the process location in relation to the target μ_0 . D is estimated as:

$$\hat{D} = \sqrt{1 + \frac{n}{n-1}\left(\boldsymbol{x} - \boldsymbol{\mu}_0\right)^T \boldsymbol{S}^{-1}\left(\boldsymbol{x} - \boldsymbol{\mu}_0\right)}$$

Both are combined to a C_{pm} value that is estimated as

$$\hat{C}_{\text{pm}} = \frac{V_{\text{tol}}}{V_{\text{proc}}} \cdot \frac{1}{\hat{D}}$$

Examples for this PCI are described in Reference.[7]

7.4.3 Type IIb

A Type IIb PCI is defined by Reference.[8] The shape of the process variation interval, which is elliptic is transformed to the shape of the tolerance region. In the case of a rectangular (cuboid or hyper-cuboid) tolerance region, this is the smallest rectangle (cuboid, hyper-cuboid) that enfolds given ellipse (ellipsoid, hyper-ellipsoid) Based on the projection intervals of the ellipse (ellipsoid, hyper-ellipsoid) of each one dimension L and U, the PCI is defined.

Examples for this PCI are described in Reference.[8]

8 Examples

8.1 Two-dimensional position tolerances

8.1.1 Type Ia Index

On a produced part, the midpoint of a drilled hole is measured. The nominal value is 80 mm in the X direction and –116,5 mm in the Y direction as shown in Figure 6. The diameter of hole is given as Ø 50 with a tolerance of ± 0,05 mm. Information on geometrical tolerances is given in ISO 1101.

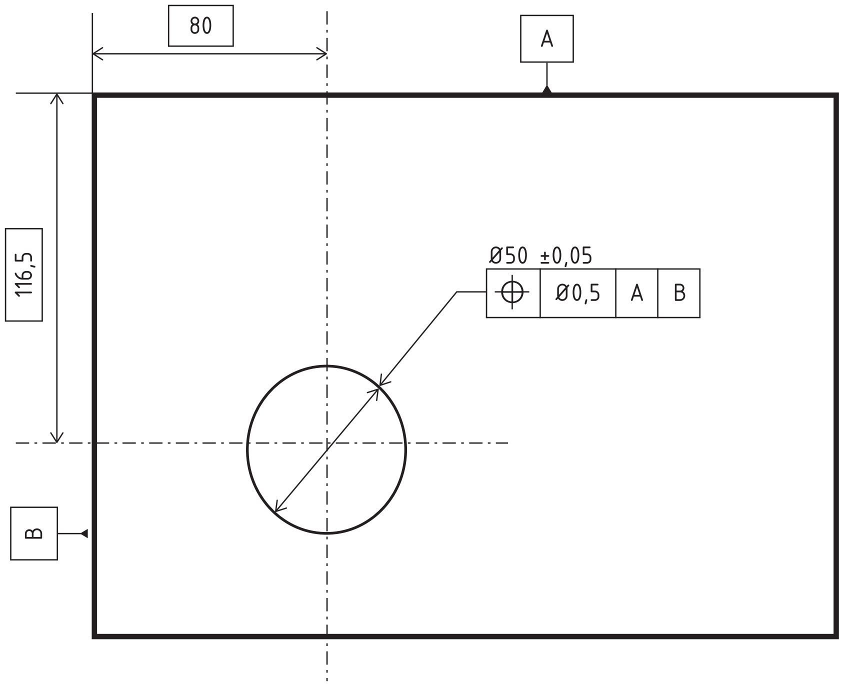


Figure 6 — Measurement task is position of a hole

One hundred sets of values from produced parts were measured (see Table 1).

The *X*- and *Y*- values were plotted in Figure 8 and the variation interval calculated. The method used to calculate the interval can be found in Annex A.

Number of measured parts $n = 100$

Specification limits: $L_X = 79{,}750$ $U_X = 80{,}250$

$L_Y = -116{,}750$ $U_Y = -116{,}250$

Table 1 — Measured values and a calculated deviation

Nr.	dev. *D*	X-coord.	Y-coord.		Nr.	dev. *D*	X-coord.	Y-coord.		Nr.	Dev *D*	X-coord.	Y-coord.
1	0,038	79,976	−116,470		36	0,090	79,995	−116,410		71	0,107	79,986	−116,394
2	0,094	79,993	−116,406		37	0,097	80,002	−116,403		72	0,073	80,016	−116,429
3	0,086	80,031	−116,420		38	0,113	80,027	−116,390		73	0,069	79,995	−116,431
4	0,041	79,968	−116,475		39	0,021	79,995	−116,520		74	0,108	79,975	−116,395
5	0,105	79,973	−116,399		40	0,085	80,010	−116,416		75	0,118	79,965	−116,387
6	0,092	79,983	−116,410		41	0,110	80,005	−116,390		76	0,122	79,971	−116,382
7	0,099	80,008	−116,401		42	0,081	80,004	−116,419		77	0,119	79,978	−116,383
8	0,086	80,014	−116,415		43	0,055	79,966	−116,457		78	0,118	79,999	−116,382
9	0,075	80,020	−116,428		44	0,097	80,013	−116,404		79	0,024	80,008	−116,477
10	0,076	79,979	−116,427		45	0,078	80,021	−116,425		80	0,094	80,005	−116,406
11	0,064	79,978	−116,440		46	0,118	79,989	−116,383		81	0,056	80,007	−116,444
12	0,086	80,016	−116,416		47	0,111	79,988	−116,390		82	0,093	80,032	−116,413
13	0,067	79,990	−116,434		48	0,057	79,987	−116,445		83	0,139	79,958	−116,368
14	0,120	79,992	−116,380		49	0,101	80,012	−116,400		84	0,122	79,990	−116,378
15	0,103	79,999	−116,397		50	0,067	80,017	−116,435		85	0,126	79,994	−116,374
16	0,119	80,016	−116,382		51	0,099	80,000	−116,401		86	0,089	80,029	−116,416
17	0,086	80,038	−116,423		52	0,101	79,995	−116,399		87	0,110	80,000	−116,390
18	0,118	80,018	−116,383		53	0,139	79,999	−116,361		88	0,084	80,010	−116,417
19	0,116	80,005	−116,384		54	0,086	80,002	−116,414		89	0,121	80,000	−116,379
20	0,118	80,071	−116,406		55	0,095	80,068	−116,433		90	0,131	79,992	−116,369
21	0,072	79,941	−116,458		56	0,103	79,990	−116,397		91	0,122	79,992	−116,378
22	0,097	79,984	−116,404		57	0,178	80,035	−116,325		92	0,062	79,990	−116,439
23	0,029	79,986	−116,475		58	0,107	79,980	−116,395		93	0,098	79,999	−116,402
24	0,093	80,043	−116,418		59	0,182	79,978	−116,319		94	0,086	79,986	−116,415
25	0,047	80,027	−116,538		60	0,099	80,000	−116,401		95	0,097	79,986	−116,404
26	0,090	80,031	−116,415		61	0,080	79,995	−116,420		96	0,092	80,020	−116,410
27	0,097	80,005	−116,403		62	0,133	79,996	−116,367		97	0,095	79,984	−116,406
28	0,122	80,024	−116,380		63	0,088	80,000	−116,412		98	0,133	79,980	−116,369
29	0,081	80,040	−116,430		64	0,107	79,948	−116,406		99	0,132	79,981	−116,369
30	0,094	80,006	−116,406		65	0,101	80,015	−116,400		100	0,058	80,033	−116,452
31	0,099	79,986	−116,402		66	0,081	79,990	−116,420					
32	0,094	79,982	−116,408		67	0,087	80,009	−116,413					
33	0,111	79,942	−116,405		68	0,067	80,004	−116,433					
34	0,135	79,975	−116,367		69	0,130	79,960	−116,376					
35	0,103	80,014	−116,398		70	0,121	80,007	−116,379					

Based on the values in Table 1, two sets of control charts were constructed.

The control chart for *x*- and *y*- coordinates show a process out-of-control. Therefore, only the process performance can be calculated.

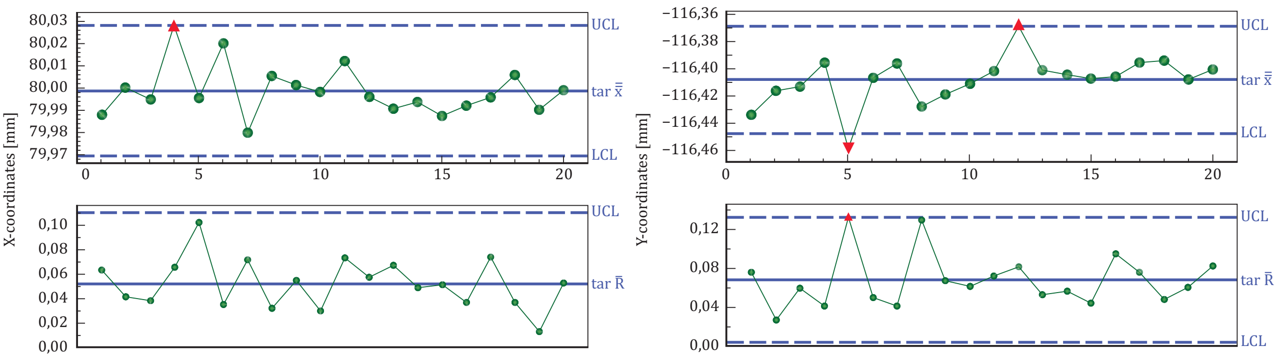


Figure 7 — $\bar{X}$ and *R* control charts for *X*-coordinate and *Y*-coordinate

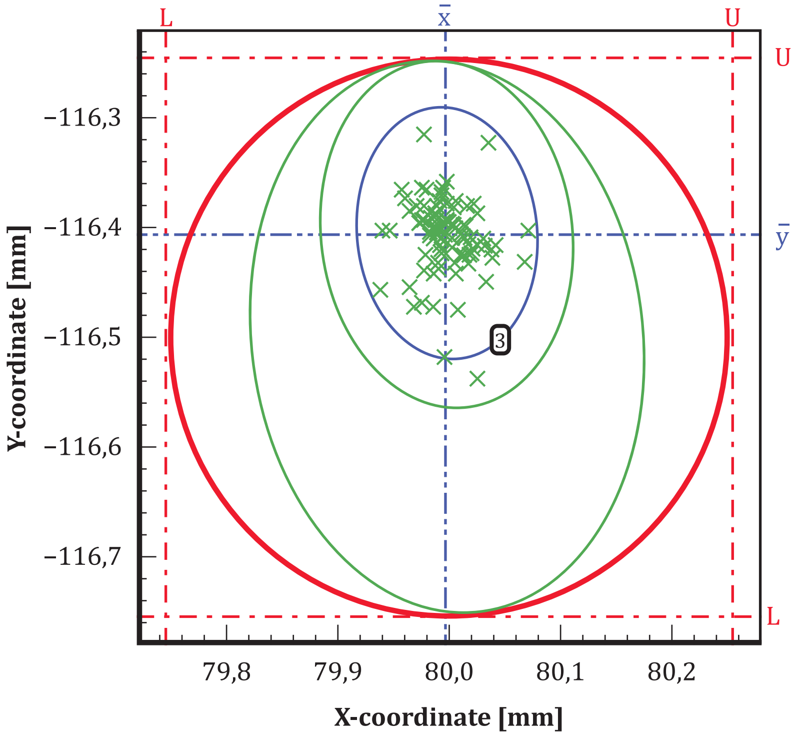


Figure 8 — Graphical presentation of the position tolerances with reference region (3) and specified tolerance

Results:

Process performance index: $\hat{P}_{\text{p}} = 2,43$

Minimum performance index: $\hat{P}_{\text{pk}} = 1,48$

The limits for the 95 % confidence interval for P_{p} are calculated to $\hat{P}_{\text{p,low}}$ = 1,99 and $\hat{P}_{\text{p,up}}$ = 2,88.

For P_{pk}:

$\hat{P}_{\text{pk,low}}$ = 1,19 and $\hat{P}_{\text{pk,up}}$ = 1,48

8.1.2 Type IIc index by calculating the capability index using the distance from target

The target location (x_0, y_0) = (80, −116,5) is specified for the centre of the hole in Figure 6. The location of each (x, y) is measured. The coordinates (x, y) denote the centre of the hole drilled. The deviation from the target location is

$$D = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

The actual calculated values for the distances D can be found in Table 1.

All deviations are plotted in a histogram shown in Figure 9. The maximum permissible deviation is 0,25 mm, because the tolerance zone is a circle centred on the target and with a diameter of 0,5 mm. The maximum permissible deviation is the radius of that circle.

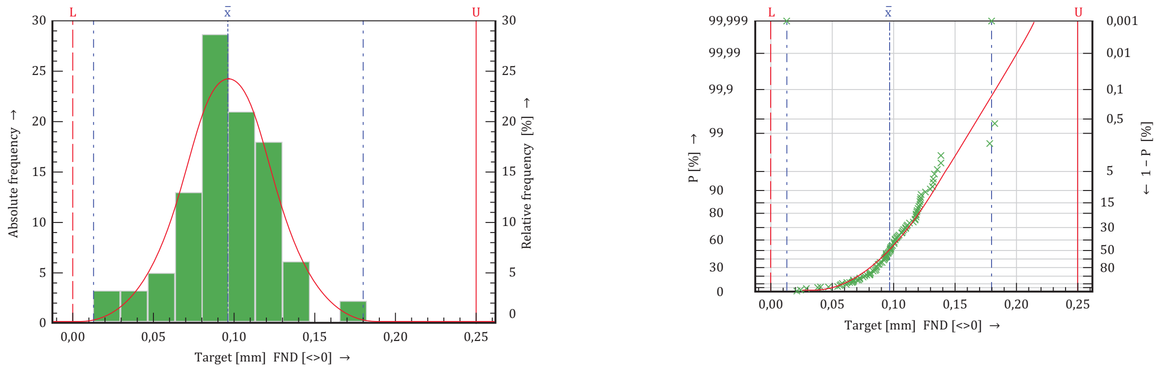


Figure 9 — Histogram and probability plot

The distribution model for the actual data set will be a Rayleigh distribution if the production is centred around the target. However, in this special case where all the values are above the target, the normal distribution fits well.

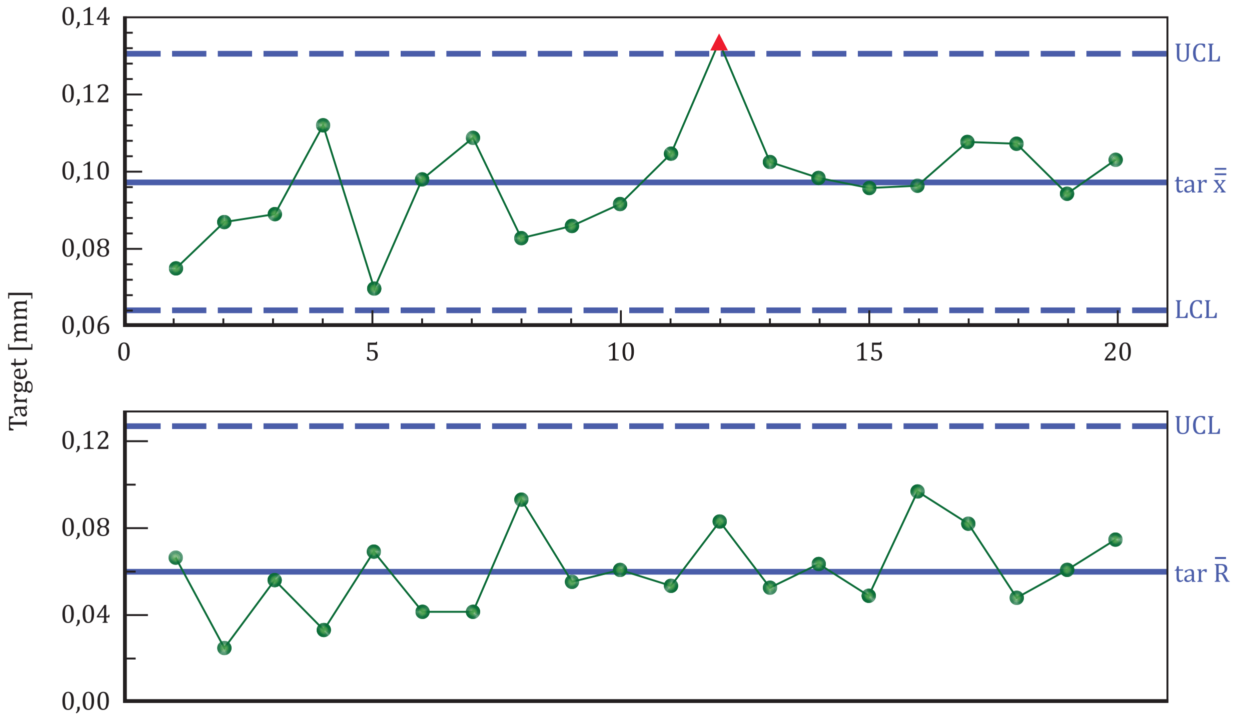


Figure 10 — Control chart

The data do not show stability in the control chart (see Figure 10). In such cases, only a probability index P_{pk} can be calculated.

Calculation of the capability indices:

— The capability index cannot be calculated because of no lower specification limit exist.

Minimum capability index:

$$\hat{C}_{PoKu} = \frac{U - D_{50\%}}{D_{99,865\%} - D_{50\%}} = \frac{0,25 - 0,096}{0,181 - 0,096} = 1,81$$

8.2 Position and dimension of a slot

A slot is to be placed in a part as shown in Figure 5. The technical function is to carry a second part in a specific position. The width of the groove is 20 ±0,2 mm. To ensure correct fitting of the parts, the position tolerance for the slot in relation to "A" is given as 0,1 mm when the dimension of the slot is at its maximum material size (19,8 mm). This maximum material condition is indicated by the symbol Ⓜ. As a result, a part can still be considered acceptable if the deviation of the slot position does not exceed the value of 0,1 mm plus the difference between the actual width of the slot and the maximum material size. The tolerance region in the situation where the maximum material condition shall not be applied is given by the rectangle in the chart of Figure 5. The extended tolerance because of MMC is shown by the right triangular extension of that rectangle.

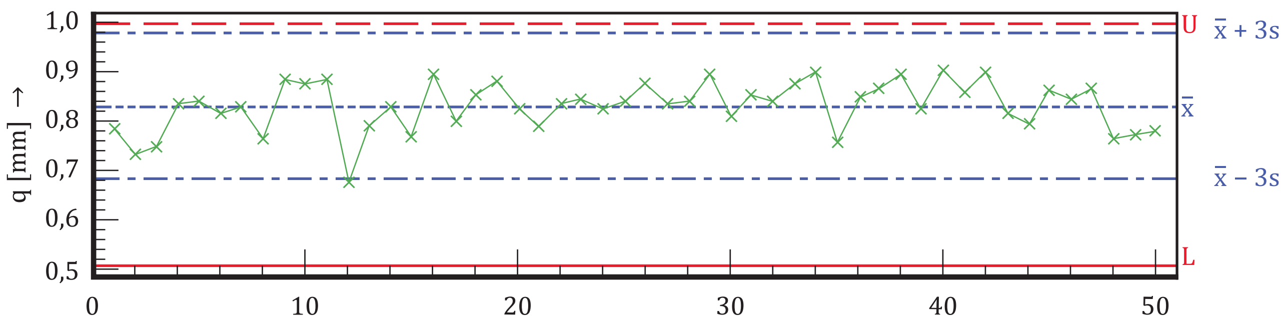
In the first step, $q(\boldsymbol{x})$ is defined for the multivariate quantity $\boldsymbol{x}$ where $\boldsymbol{x}_1$ = width and $\boldsymbol{x}_2$ = position. The function chosen is composed of three linear functions q_i, i = 1, 2, 3 of the type $q_i = a_{1i}\boldsymbol{x}_1 + a_{2i}\boldsymbol{x}_2 + a_{0i}$. Their coefficients are set to appropriate values in order to give values of $q_{max} = 1$ in the target and $q_{bound} = 0,5$ at the tolerance limits.

Table 2 — Measured values and the calculated qualification values

No.	q	Width mm	Position mm		No.	q	Width mm	Position mm		No.	q	Width mm	Position mm
1	0,744	20,102	0,06		18	0,828	20,069	0,09		35	0,862	20,033	0,116
2	0,845	20,062	0,11		19	0,899	20,04	0,091		36	0,879	20,027	0,1
3	0,858	20,016	0,102		20	0,807	20,007	0,123		37	0,671	20,131	0,074
4	0,846	20,035	0,127		21	0,838	20,065	0,083		38	0,865	20,026	0,107
5	0,829	20,068	0,075		22	0,781	20,087	0,071		39	0,897	20,035	0,097
6	0,906	20,038	0,091		23	0,866	20,053	0,09		40	0,777	20,001	0,135
7	0,763	20,002	0,144		24	0,888	20,045	0,096		41	0,771	20,009	0,146
8	0,789	20,084	0,074		25	0,897	20,041	0,102		42	0,823	20,026	0,132
9	0,798	20,001	0,122		26	0,768	20,093	0,084		43	0,84	20,012	0,108
10	0,797	20,081	0,069		27	0,824	20,07	0,084		44	0,832	20,01	0,111
11	0,841	20,063	0,09		28	0,85	20,06	0,116		45	0,875	20,05	0,112
12	0,874	20,05	0,085		29	0,825	20,07	0,091		46	0,884	20,031	0,101
13	0,731	20,108	0,068		30	0,843	20,063	0,097		47	0,894	20,042	0,091
14	0,792	20,083	0,076		31	0,843	20,063	0,114		48	0,852	20,018	0,106
15	0,763	20,095	0,081		32	0,818	20,073	0,07		49	0,832	20,025	0,126
16	0,888	20,045	0,081		33	0,9	20,036	0,096		50	0,854	20,058	0,091
17	0,815	20,008	0,119		34	0,755	19,992	0,139					

In the second step, process data for $\boldsymbol{x}$ is acquired. For demonstration purposes, 50 values are considered to be taken from a milling process as an example. The measured values for width and position are summarized in Table 2. For each of the points, the corresponding value of the qualification function $q(x_1, x_2)$ is calculated. The data are illustrated in the chart of Figure 5 as well as in the individual value charts in Figure 11.

From Figure 5, it can be seen that the width and the deviation in position are correlated. Higher values for the width are found together with lower values for the position. The lower charts in Figure 11 show that the process starts at higher values for the widths and in-spec.-values for the position. Due perhaps to tool wear or other systematic influences, the values of the widths decrease and the values for position increase. The latter ones exceed the 0,1 mm tolerance limit and make use of the bonus tolerance due to the maximum material condition. The upper chart in Figure 11 shows the course of the q-values. These are slightly rising as the process vector moves in the direction of the target. They fall again as the tolerance limit is approached.



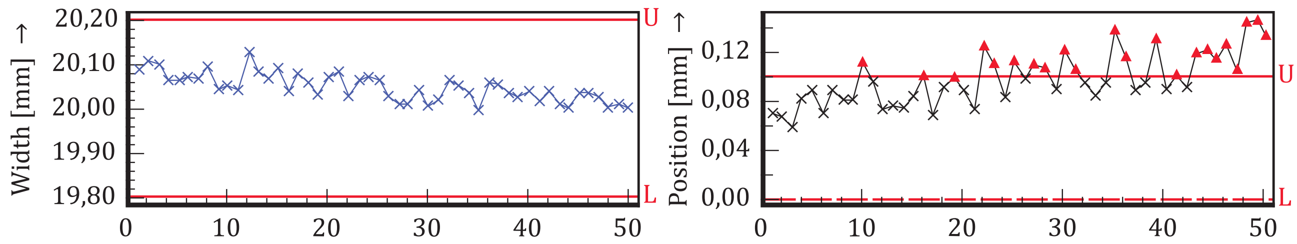


Figure 11 — Individual value charts for the qualification values (upper graph), the values for the width (lower left graph) and the position (lower right graph)

In the third step, the distribution of the *q*-values is identified. Although the distribution of width and position is probably not normal because of trends, a distribution for the values *q* can be found. Figure 12 shows the identified Pearson-type distribution density function.

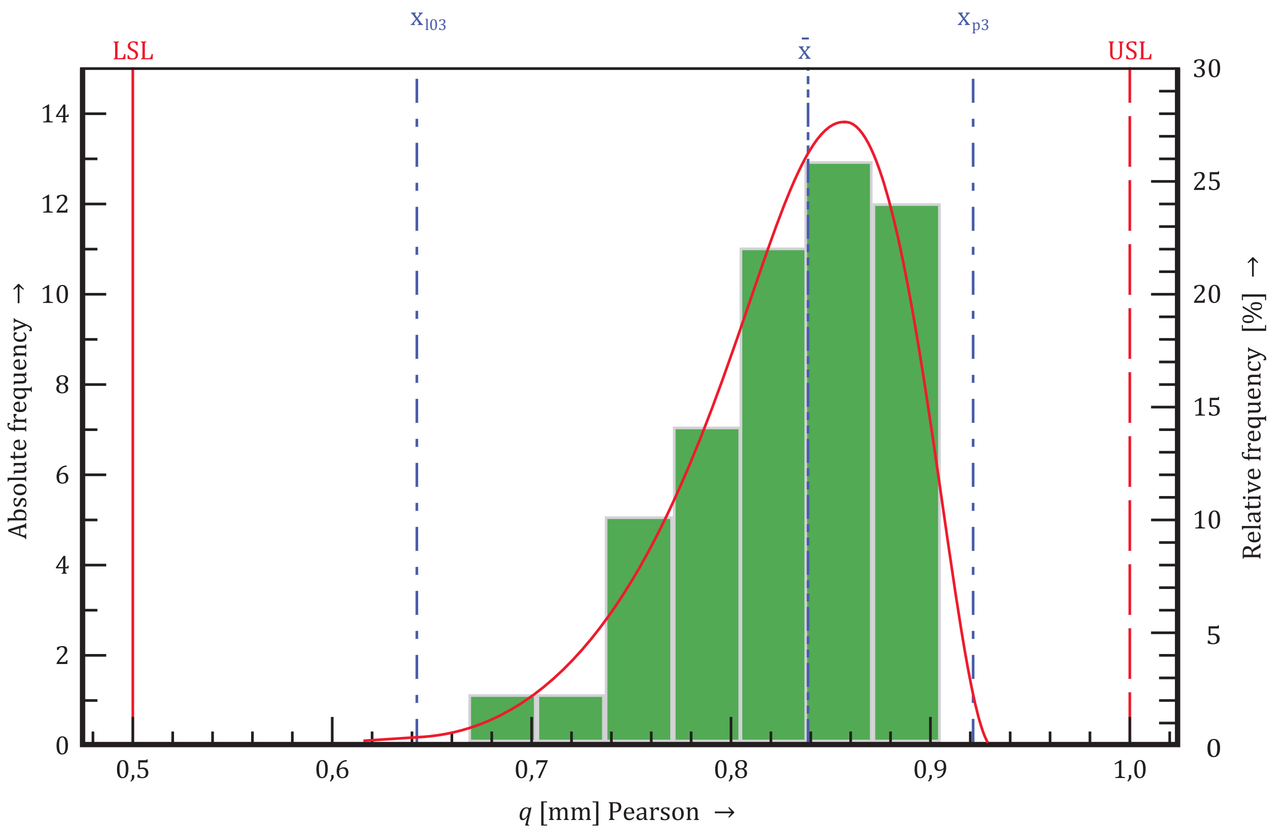


Figure 12 — Histogram and distribution density function of the identified distribution for *q*

Based on that and on the q_{bound} = 0,5 tolerance limit, the Type II - PCI can be estimated as follows:

Type IIc capability index:

$$\hat{P}_{\text{pk}} = \frac{\hat{q}_{50\ \%} - q_{\text{bound}}}{\hat{q}_{50\ \%} - \hat{q}_{0,135\%}} = \frac{0,837\ 5 - 0,5}{0,837\ 5 - 0,641\ 4} = 1,72$$

Since the number of measured values is too small to judge if the process is in a state of statistical control, the performance index P_{pk} is used instead of the capability index.

To calculate the Type I - PCI, the percentage of conform parts are to be calculated. For the identified Pearson distribution, a defect rate is 0,01 ppm.

Type Ic capability index:

$$\hat{P}_{\text{pk}} = \frac{1}{3}\Phi^{-1}\left(\frac{0,999\ 999\ 990 + 1}{2}\right) = \frac{5,73}{3} = 1,91$$

Alternatively, in the case that reciprocity requirement for the width could may be applied (by adding ® behind the dimension for the width in the drawing), the expression for q_i can be simplified. By an additional monitoring for the width, one can use the expression $q = x_1$ - 19,7 - x_2 for a definition of the qualification function. In that case, q_{bound} = 0 applies and a conforming part has $q > 0$.

$$\breve{P}_{\text{pk}} = \frac{\hat{X}_{50\%} - L}{\hat{X}_{50\%} - \hat{X}_{0,135\%}} = \frac{0,249 - 0}{0,249 - 0,097} = 1,64$$

Annex A
(informative)

Derivation of formulae

A.1 Useful properties of the multivariate normal distribution in calculating capability indices

The density of the d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{d}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} (\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}-\boldsymbol{\mu}) \right), \quad x \in R^d,$$

if the covariance matrix $\boldsymbol{\Sigma}$ is positive definite such that its inverse $\boldsymbol{\Sigma}^{-1}$ exists. Here $\boldsymbol{X}$ and $\boldsymbol{\mu}$ are d-dimensional vectors and $\boldsymbol{\Sigma}$ is a $d \times d$ matrix. Vectors are column vectors and T denotes the transpose of a matrix or a vector, i.e. $\boldsymbol{x}^T$ is $\boldsymbol{X}$ written as a row vector. The d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is denoted by $N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

The contours of constant density are

$$\left\{ \boldsymbol{x} \middle| (\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}-\boldsymbol{\mu}) = c^2 \right\}$$

and they are intervals for $d = 1$, ellipses for $d = 2$ and ellipsoids for $d \geq 3$.

The probability that a process value lies within the area bounded by the contour ellipsoids can be calculated from the χ^2 distribution on d degrees of freedom. If $\boldsymbol{X}$ follows a d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, then

$$P\left((\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{X}-\boldsymbol{\mu}) \leq c^2 \right) = F_{\chi^2(d)}\left(c^2\right)$$

where $F_{\chi^2(d)}$ denotes the distribution function of the χ^2 distribution on d degrees of freedom.

It follows that the contour ellipsoid

$$\left\{ \boldsymbol{x} \middle| (\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}-\boldsymbol{\mu}) = \left(\sqrt{F_{\chi^2(d)}^{-1}(p)} \right)^2 \right\}$$

is the boundary of an region with a probability of p. Here, $F_{\chi^2(d)}^{-1}(p)$ is the p-fractile of the χ^2 distribution on d degrees of freedom. This quantity is sometimes denoted by $\chi_p^2(d)$.

If $X_1,\ldots,X_n$ is a sample from a d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, then $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated as

$$\boldsymbol{\mu} \leftarrow \hat{\boldsymbol{\mu}} = \bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i$$

and

$$\boldsymbol{\Sigma} \leftarrow \boldsymbol{S} = \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \bar{\boldsymbol{x}}\right)\left(\boldsymbol{x}_i - \bar{\boldsymbol{x}}\right)^T$$

where "$\leftarrow$" is read as "is estimated by."

A.2 Motivation for the definitions of multivariate capability

Consider first the C_{p} index in the one-dimensional case. The tolerance interval is the interval [L, U]. Let $X \sim N(\mu, \sigma^2)$ with $\mu = (U + L)/2$, i.e. the distribution is centred on the midpoint of the tolerance interval. The probability that a process value lies inside the tolerance interval

$$P = P_{N((U+L)/2,\sigma^2)}\left(L < X < U\right) = P_{N((U+L)/2,\sigma^2)}\left(L - \frac{L+U}{2} < X - \frac{L+U}{2} < U - \frac{L+U}{2}\right)$$

$$= P_{N((U+L)/2,\sigma^2)}\left(\frac{L-U}{2\sigma} < \frac{X - \frac{L+U}{2}}{\sigma} < \frac{U-L}{2\sigma}\right) = \Phi\left(\frac{U-L}{2\sigma}\right) - \Phi\left(-\frac{U-L}{2\sigma}\right)$$

$$= \Phi\left(\frac{U-L}{2\sigma}\right) - \left[1 - \Phi\left(\frac{U-L}{2\sigma}\right)\right] = 2\Phi\left(\frac{U-L}{2\sigma}\right) - 1 = 2\Phi\left(3C_{\mathrm{p}}\right) - 1$$

It follows that

$$C_{\mathrm{p}} = \frac{1}{3}\Phi^{-1}\left(\frac{P+1}{2}\right)$$

where P is the probability that a process value lies inside the tolerance interval for a normal distribution centred on the midpoint of the tolerance interval and with variance σ^2.

Consider next the C_{pk}, $C_{\mathrm{pk}U}$ and the $C_{\mathrm{pk}L}$ indices again in the one-dimensional case. Now, consider a normal distribution with mean μ and variance σ^2 and assume that μ is larger than $(U+L)/2$ but smaller than U, i.e. μ is in the tolerance interval but μ is closer to the upper limit of the tolerance interval than to the lower limit. Now, consider the probability of the largest interval that is centred on the mean and contained in the tolerance interval. This is the interval [$2\mu - U$, U].

$$P = P_{N(\mu,\sigma^2)}\left(2\mu - U < X < U\right) = P_{N(\mu,\sigma^2)}\left(2\mu - U - \mu < X - \mu < U - \mu\right)$$

$$= P_{N(\mu,\sigma^2)}\left(\frac{\mu - U}{\sigma} < \frac{X - \mu}{\sigma} < \frac{U - \mu}{\sigma}\right) = \Phi\left(\frac{U - \mu}{\sigma}\right) - \Phi\left(-\frac{U - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{U - \mu}{\sigma}\right) - \left[1 - \Phi\left(\frac{U - \mu}{\sigma}\right)\right] = 2\Phi\left(\frac{U - \mu}{\sigma}\right) - 1 = 2\Phi\left(3C_{\mathrm{pk}U}\right) - 1$$

It follows that

$$C_{\mathrm{pk}U} = \frac{1}{3}\Phi^{-1}\left(\frac{P + 1}{2}\right)$$

When μ is larger than $(U+L)/2$, then $C_{\mathrm{pk}U}$ is smaller than $C_{\mathrm{pk}L}$ and

$$C_{\mathrm{pk}} = \min\{C_{\mathrm{pk}U}, C_{\mathrm{pk}L}\} = C_{\mathrm{pk}U}$$

If μ lies between L and $(U+L)/2$, then similar calculations give

$$C_{\mathrm{pk}L} = \frac{1}{3}\Phi^{-1}\left(\frac{P + 1}{2}\right)$$

and

$$C_{\mathrm{pk}} = \min\{C_{\mathrm{pk}U}, C_{\mathrm{pk}L}\} = C_{\mathrm{pk}L}$$

In both cases, the formula for C_{pk} is:

$$C_{\mathrm{pk}} = \frac{1}{3}\Phi^{-1}\left(\frac{P + 1}{2}\right)$$

P is still the probability of the largest interval that is centred on the mean and contained in the tolerance interval, but in this case the interval is [L, $2\mu - L$].

We need to consider the situation where μ lies outside the tolerance interval. Suppose μ is larger than U. We still wish to calculate the probability of the interval that is centred on μ and extends from the boundary of the tolerance interval that is closest to μ. In this case, the interval is $[U, 2\mu - U]$.

$$P = P_{N(\mu,\sigma^2)}\left(U < X < 2\mu - U\right) = P_{N(\mu,\sigma^2)}\left(U - \mu < X - \mu < 2\mu - U - \mu\right)$$
$$= P_{N(\mu,\sigma^2)}\left(\frac{U-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\mu-U}{\sigma}\right) = \Phi\left(\frac{\mu-U}{\sigma}\right) - \Phi\left(\frac{U-\mu}{\sigma}\right)$$
$$= 1 - \Phi\left(\frac{U-\mu}{\sigma}\right) - \Phi\left(\frac{U-\mu}{\sigma}\right) = 1 - 2\Phi\left(\frac{U-\mu}{\sigma}\right) = 1 - 2\Phi\left(3C_{\mathrm{pk}U}\right)$$

It follows that

$$C_{\mathrm{pk}U} = \frac{1}{3}\Phi^{-1}\left(\frac{1-P}{2}\right)$$

In this case, $C_{\mathrm{pk}U}$ is smaller than $C_{\mathrm{pk}L}$ and

$$C_{\mathrm{pk}} = \min\{C_{\mathrm{pk}U}, C_{\mathrm{pk}L}\} = C_{\mathrm{pk}U}$$

so the formula applies to C_{pk}, i.e.

$$C_{\mathrm{pk}} = \frac{1}{3}\Phi^{-1}\left(\frac{1-P}{2}\right)$$

The idea behind extending this to a definition of a capability indices for the multivariate normal distribution is to replace intervals by ellipses for the two-dimensional normal distribution and ellipsoids for the d-dimensional normal distribution when $d \geq 3$.

Consider a d-dimensional normal distribution with covariance matrix $\boldsymbol{\Sigma}$. In order to calculate the multivariate C_{p} index, centre the normal distribution to have the mean at the centre of the tolerance zone. For that normal distribution, determine the largest contour ellipsoid that is completely contained in the tolerance zone and determine the probability of the volume bounded by that contour ellipse under the d-dimensional normal distribution with covariance matrix $\boldsymbol{\Sigma}$ and mean at the centre of the tolerance zone. Denote that probability by P. Then, the multivariate C_{p} index is

$$C_{\mathrm{p}} = \frac{1}{3}\Phi^{-1}\left(\frac{P+1}{2}\right)$$

In order to estimate a C_{p} index from the d-dimensional data, start by estimating the covariance matrix of the multivariate normal distribution from the data. Denote the estimate by $\hat{\Sigma}$ and use that covariance matrix to determine the contour ellipsoid and its probability $\hat{P}$. Finally, the estimated multivariate C_{p} index is

$$\hat{C}_{\mathrm{p}} = \frac{1}{3}\Phi^{-1}\left(\frac{\hat{P}+1}{2}\right)$$

Calculation of the C_{pk} index involves both the mean and the variance of the distribution, so consider a d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. For the $N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution, calculate the largest contour ellipsoid that is completely contained in the tolerance zone, if $\boldsymbol{\mu}$ is contained in the tolerance zone, or the largest contour ellipsoid that is not contained in the tolerance zone, if $\boldsymbol{\mu}$ is not contained in the tolerance zone. Now, the probability, P, of the volume contained in the contour ellipsoid under the $N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution is calculated. Finally, the C_{pk} index is calculated as:

$$C_{\text{pk}} = \frac{1}{3}\Phi^{-1}\left(\frac{P+1}{2}\right)$$

if $\boldsymbol{\mu}$ is in the tolerance zone and as

$$C_{\text{pk}} = \frac{1}{3}\Phi^{-1}\left(\frac{1-P}{2}\right)$$

if $\boldsymbol{\mu}$ is not in the tolerance zone.

In order to estimate a C_{pk} index from the d-dimensional data, start by estimating the mean and the covariance matrix of the multivariate normal distribution from the data. Denote the estimates by $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$. Now, the estimated C_{pk} index is calculated as above but using the $N_d\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}\right)$ distribution. The formulae read

$$\hat{C}_{\text{pk}} = \frac{1}{3}\Phi^{-1}\left(\frac{\hat{P}+1}{2}\right)$$

if $\hat{\boldsymbol{\mu}}$ is in the tolerance zone and

$$\hat{C}_{\text{pk}} = \frac{1}{3}\Phi^{-1}\left(\frac{1-\hat{P}}{2}\right)$$

if $\hat{\boldsymbol{\mu}}$ is outside the tolerance zone.

Annex B
(informative)

Shaft imbalance example

B.1 Imbalance example

The measuring size imbalance describes the deviations in the rotor's distribution of the mass from the ideal mass distribution with basis in the figure axis of the rotor. The axial distribution of imbalance is assessed from the indication of the imbalance in relation to two different planes. The choice of the current rotor plan is random. However, it is important for the measurement of the process capability that the indication of the imbalance tolerance and residual imbalance value is indicated from the same plans. For more comprehensive definitions and explanations of the size of the imbalance, refer to extensive explanations in the standard literature. On the basis of these documents, the technical terms are used here without further definitions.

The number of the different rotors being aligned is extremely multitudinous. The statistical methods described in this document are in principle, independent of the rotor size. However, certain rotor sizes and accelerations and brake courses are very time consuming and, in these cases, you can make repeated measurements at constant revolutions of the rotor.

A speciality of the measurement size imbalance is the bivariate quantities. In this example, the calculations of the process capability are described. The objective of the alignment process is the limitation of the residual imbalance which is given by the tolerance area under consideration of the agreed capability indices. The distribution of the residual imbalance contains both random and systematic failure influences. The difference between these observed deviations of the bivariate size and tolerance area as demonstrated in this part of ISO 22514 by using PCI.

As a consequence, it is not possible to unambiguously distinguish between a correct or wrong calculation in a bivariate or univariate form. Therefore, the bivariate variant is preferred. In both the described exceptional cases, the univariate calculation best corresponds to the reality.

However, it is important that the parties participating in the process assessment agree on a sensible use of the described assessment methods in advance.

The indices appearing in this way are scaled on the basis of the experiences with process capability of univariate sizes. Even though the capability indices maintain the same designation, a quality requirement of e.g. 1,67 means, without comparison, a larger requirement for a process with bivariate measuring values than a corresponding one with univariate measuring values.

An additionally important quantity with influence on statistical assessment of alignment processes is the presence or non-presence of a rotor-internal angle system. Electro-anchors for instance often do not have a notation of their own angle system. In contrast, an unambiguous angle system exists for crankshafts, which is well fitted for use. In the first case, a repetition of the imbalance measurements usually shows the same contribution to the imbalance, without errors, though it is another angle each time. Furthermore, the given circumstances during the production course of the rotor can entail that the precondition (that a bivariate normal distribution appears) is not fulfilled for the calculations described here.

The number of the different rotors being aligned is extremely multitudinous. The statistical methods described in this part of ISO 22514 are, in principle, independent of the rotor size. However, certain rotor sizes and accelerations and brake courses are very time consuming, and in these cases, you can make repeated measurements at constant revolutions of the rotor.

B.2 Example of an examination of capabilitiy indices

The following example shows an examination of capability indices when approving an alignment machine used for aligning crankshafts. The stability of the process is demonstrated by using a control chart.

The measuring of imbalance happens at two measuring levels.

The specification limit for imbalance is 140 gmm.

A sample of n = 40 crankshafts is aligned on the machine.

The residual imbalances obtained in Table B.1 are available. They are graphically presented in Figures B.1 and B.2.

Table B.1 — Single values

Level 1				Level 2		
Value no.	***X*-Coord.**	***Y*-Coord.**		**Value no.**	***X*-Coord.**	***Y*-Coord.**
1	0,885	12,604		1	−19,324	23,276
2	20,068	178,425		2	12,318	−169,764
3	−5,521	6,054		3	−9,016	1,470
4	1,476	2,838		4	−6,814	−1,508
5	1,455	−11,472		5	−14,592	9,731
6	1,794	0,567		6	−34,272	13,383
7	30,136	41,994		7	−36,690	1,563
8	−45,382	9,398		8	12,357	8,047
9	−35,158	−3,739		9	20,696	4,372
10	−3,660	−2,120		10	3,090	3,458
11	−25,347	5,964		11	3,326	−7,789
12	−7,926	1,247		12	−0,222	6,666
13	−44,010	−5,069		13	40,998	2,136
14	−10,365	−9,795		14	−11,033	11,969
15	18,726	−1,139		15	−30,559	−2,421
16	−9,878	−6,160		16	−12,540	21,162
17	−24,537	−14,385		17	17,230	12,795
18	9,477	4,240		18	−20,771	7,762
19	−9,621	−6,873		19	−4,295	5,838
20	−24,828	−14,058		20	0,373	8,879
21	−24,426	3,781		21	7,877	19,057
22	22,463	2,530		22	−17,939	14,573
23	10,189	−29,127		23	−26,314	36,687
24	−8,204	−6,575		24	0,147	9,822
25	−14,549	−3,084		25	22,519	−8,161
26	5,275	−18,891		26	−24,116	32,627
27	−2,593	−9,904		27	23,783	16,612
28	−9,663	−3,303		28	−8,201	−3,193
29	4,277	−7,275		29	−10,786	6,597
30	16,190	−0,354		30	−36,605	0,287

Table B.1 *(continued)*

Level 1				Level 2		
Value no.	***X*-Coord.**	***Y*-Coord.**		**Value no.**	***X*-Coord.**	***Y*-Coord.**
31	−13,753	1,395		31	−0,683	19,652
32	−26,941	4,794		32	22,566	−1,902
33	14,465	−8,385		33	−17,977	−0,198
34	−3,879	−6,028		34	−5,684	4,800
35	−51,059	−7,822		35	46,300	7,634
36	10,574	−7,357		36	−26,883	−7,430
37	0,311	−5,701		37	1,859	19,344
38	−12,551	10,266		38	0,909	12,527
39	−3,687	−2,249		39	6,882	21,786
40	3,833	6,598		40	−5,863	7,641

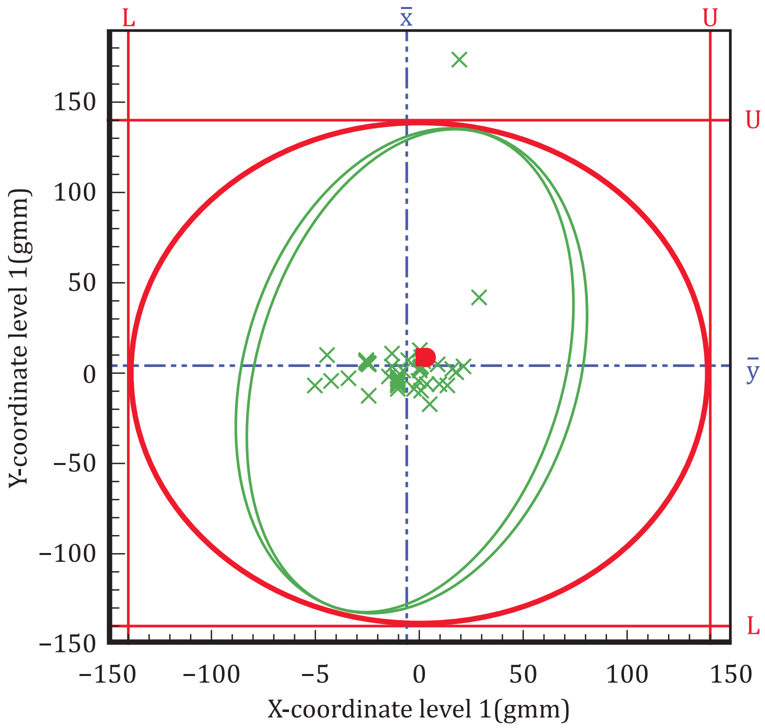


Figure B.1 — Residual imbalance level 1 — Graphical presentation of the imbalance with random variation interval and tolerance

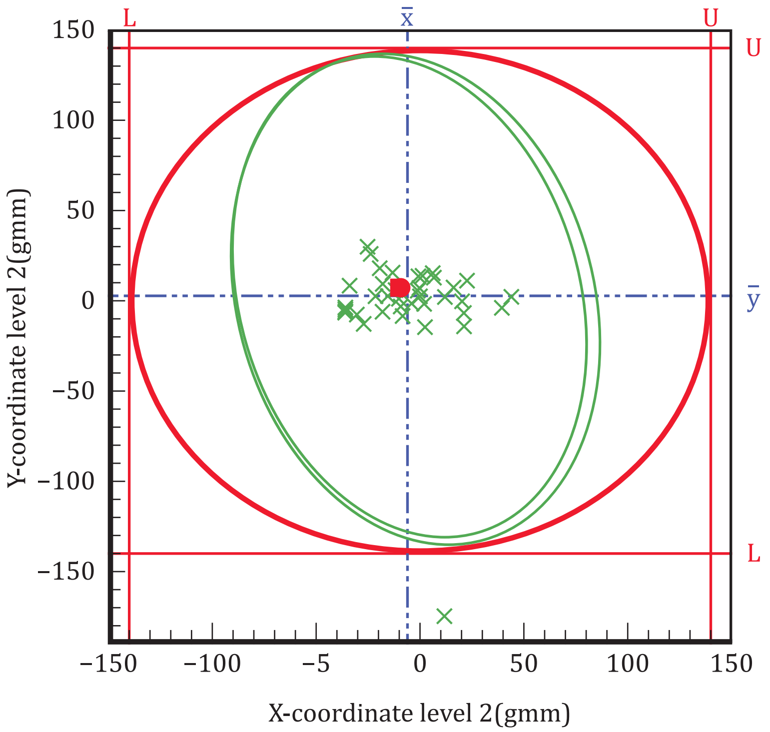


Figure B.2 — Residual imbalance level 3 — Graphical presentation of the imbalance with random variation interval and tolerance

Results:

Level 1:	Capability index:	1,37
	Minimum capability index:	1,36
Level 2:	Capability index:	1,41
	Minimum capability index:	1,36

Annex C
(informative)

Hole position example

C.1 Numerical example distance from target

Consider the process of drilling a hole in a parts production process. A target location (x_0, y_0) is specified for the centre of the hole. The location (x_0, y_0) is measured from some specified reference point. The coordinates (x, y) denote the centre of the hole drilled. Then, the position deviation for the hole location is defined as *D.*

$$D = \sqrt{(x - x_0)^2 + (y - y_0)^2} \qquad \text{(C.1)}$$

(ISO 1101). In such a case, the specification limit is usually given as a circle with the diameter of D_0 and the centre of (x_0, y_0) . Therefore, the quantity *D* can be used as the process quantity of process capability for quality assurance.

The quantity *D* can be regarded as a one–dimensional proxy for the two-dimensional quantity of the realized hole location.

Two bivariate normal distributions are supposed as shown in Figure C.1. These distributions are regarded as the distributions of processing location of two processes, Process A and Process B, respectively. (See Figure C.1.) Their covariance matrices are

$$\Sigma_A \begin{bmatrix} \frac{r^2}{9} & 0 \\ 0 & \frac{r^2}{9} \end{bmatrix}$$

$$\Sigma_B \begin{bmatrix} \frac{r^2}{9} & 0 \\ 0 & \delta^2 \end{bmatrix}$$

where δ is small.

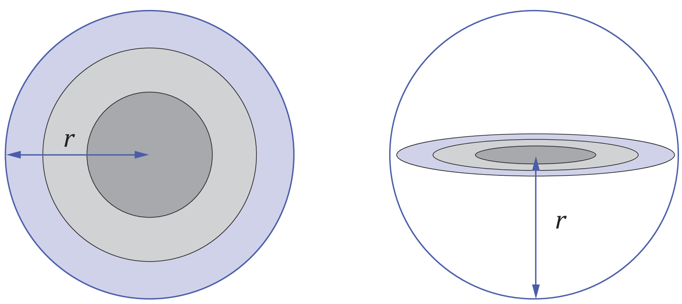


Figure C.1 — Two distributions of Process A (left figure) and Process B (right figure)

From Figure C.1, it is definite that Process B is more capable than Process A. However,

$$Var\left(D_{\text{A}}\right)=\frac{r^2}{9}\left(2-\frac{\pi}{2}\right),\ Var\left(D_{\text{B}}\right)=\frac{r^2}{9},$$

hence, Process A is more capable than Process B judging from the standard deviation of *D*.

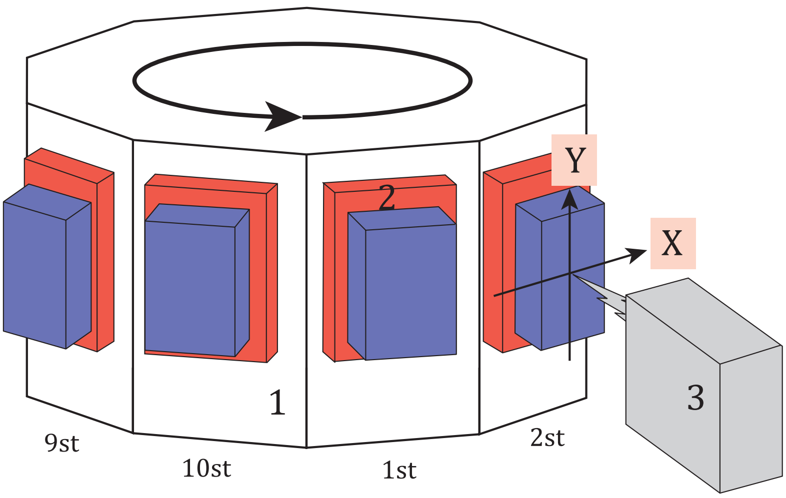
The quantity *D* is definitely a quality to ensure. This numerical example shows that a quantity to ensure is not necessarily identical to a quantity to control.

C.2 Practical example

Consider the following parts production process by a transfer machine, which is a turntable system in the horizontal axis direction (the *x*-coordinate direction) as shown in Figure C.2 In this process, drilling holes, cutting sections, tapping screws and so on are performed at each station in turn. The subject of this case study is drilling holes.

The subject part is shown in Figure C.3. The quantity to be ensured is the location of a hole, which is the geometrical quantity *D* shown in Formula C.1. Its specification region, in this case, is a circle with a diameter of 0,1 mm.

Figure C.4 shows the histogram of the value *D*. Its sample size is 50. The process capability index C_{p} is 1,35. The index C_{p} indicates that the level of process capability is high. However, when the production stage proceeded to mass production, a nonconforming problem appeared. Although the process capability had been highly acceptable, why did this problem appear?



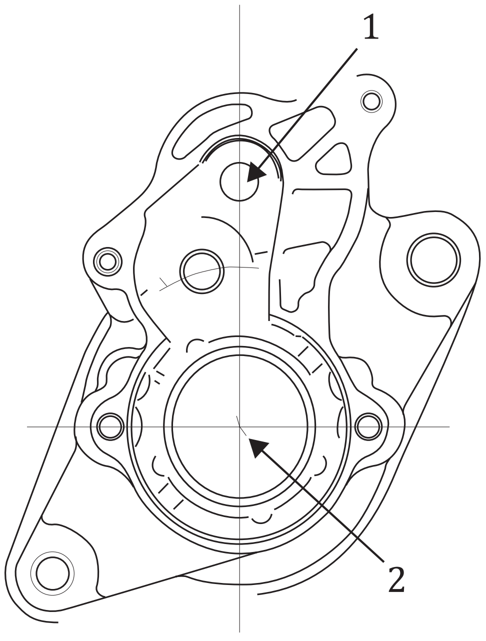
Key

1 turntable

2 jig

3 processing unit

Figure C.2 — Turntable mechanism of transfer machine



Key

1 hole position

2 point of reference

Figure C.3 — Rough sketch of the part

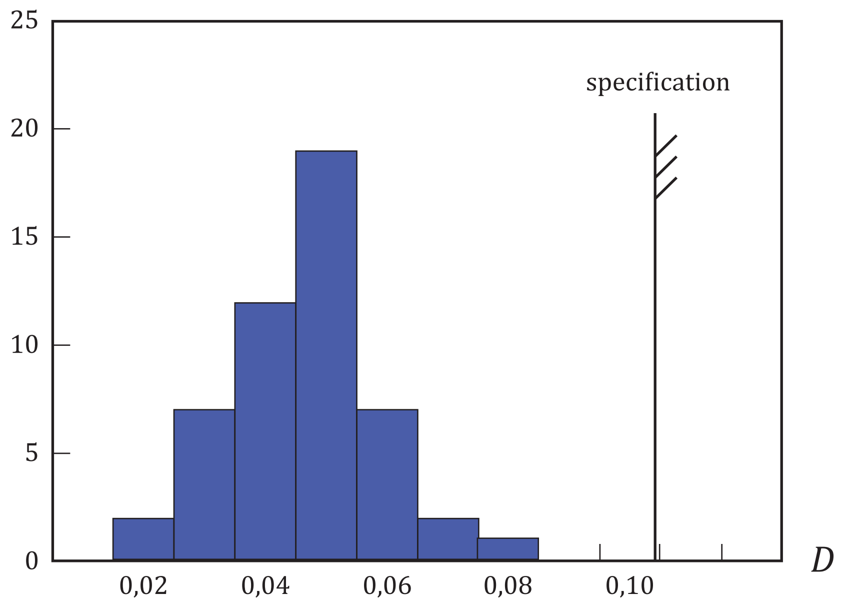


Figure C.4 — Histogram of geometrical quantity *D*

Figure C.4 shows two-dimensional plots of the centre of the hole drilled, where the origin of the coordinates is the target location (x_0, y_0). Figure C.5 indicates that the variability for the *x*-coordinate is much larger than that for the *y*-coordinate. The standard deviations of the location for *x*-coordinate and *y*-coordinate are denoted as s_x and s_y, respectively. Then $s_x = 0{,}0177$ and $s_y = 0{,}0050$. As a result, the process capabilities, $C_p(x)$ and $C_p(y)$, are obtained as $C_p(x)=0{,}94$ and $C_p(y)=3{,}35$. The process capability for the *x*-coordinate is much less than that for the *y*-coordinate.

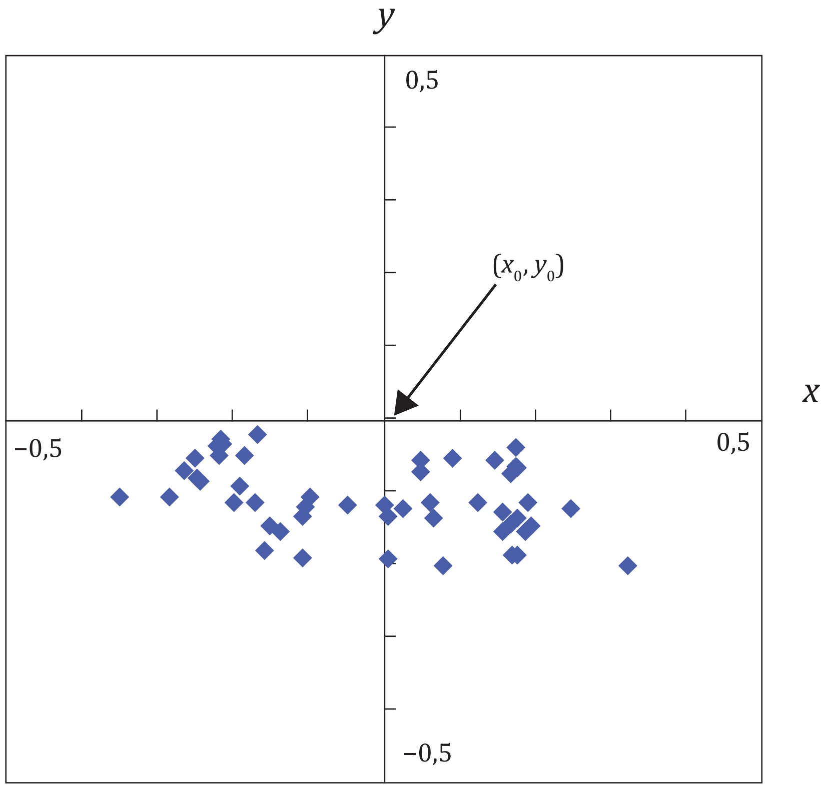


Figure C.5 — Two-dimensional plots of the centre of the produced hole

Some causes of location variability can be listed: the variability of the location of the jig, for setting the part and so on; however, the above result is caused by the turntable type mechanism of the transfer machine (Figure C.2). Consequently, it is confirmed that the low process capability is due to the variability of the turntable stopping position, which reveals the variability in the *x*-coordinate direction, and it is a serious common cause in the transfer machine.

Actually, selecting the quantity *D* as a control quantity of this process resulted in failure.

Annex D
(informative)

Construction of the quality function

Construction of the quality function $q(\underline{x})$ for the example is given in 8.2.

First, a set of limiting functions $g_i(\underline{\Delta x})$ are defined with i = 1(1)M. These functions define the tolerance region. One function is given for one tolerancing restriction. A target point $\underline{x}_{target}$ with $\underline{q}(\underline{x}_{target}) = \underline{q}_{max} = 1$ is given by the specification. The deviation from the target vector $\underline{x}_{target}$ (difference from the measured values to the target value in each dimension) is calculated and denoted as $\underline{\Delta x}$.

$$\underline{\Delta x} = (\underline{x}_{1\ target} - \underline{x}_1, \underline{x}_{2\ target} - \underline{x}_2, \dots)^T .$$

The limiting functions $\underline{g}_i(\underline{\Delta x})$ (M = 3 in the example) are chosen to be linear functions of the form:

$$g_i(\underline{\Delta x}) = \underline{b}_i{}^T \cdot \underline{\Delta x} + c_i,$$

where $\underline{b}_i$ is the vector of slopes and c_i is a constant in the $\underline{i}$-th function.

For each deviation, vector $\underline{\Delta x}$, an axis $\underline{\Delta x t}$ is defined where t is a parameter $\underline{t} \in [0,+\infty)$. The axis starts with $\underline{t} = 0$ in the target point $\underline{x}_{target}$. With increasing $\underline{t}$, the axis extends until crossing the limiting functions g_i. For all $\underline{I}$ functions limiting function, the parameter $\underline{t}_{\mathbf{i}}$ can be calculated that fulfils:

$$0 = (\underline{b}_{\mathrm{i}}{}^{\mathrm{T}} \cdot \underline{\Delta x})\underline{t}_i + \underline{c}_i,$$

Among all $\underline{t}_i$, the minimum, $\underline{t}_{min}$, is found.

$$\underline{t}_{min} = \min \{\underline{t}_i / \underline{t}_i \geq 0\}$$

From $\underline{t}_{min}$, a value $\underline{a}(\underline{x})$ is calculated that describes the difference from the target:

$$\underline{a}(\underline{x}, \underline{x}_{target}) = 1/\underline{t}_{min}(\underline{\Delta x}, g_i(\underline{\Delta x}))$$

Thus, $\underline{q}(\underline{x})$ can be scaled to $\underline{q}_{max} = 1$ and $\underline{q}_{bound} = 0{,}5$ using the linear function:

$$q(x) = \begin{cases} 1-(X)/2 & \text{if } a(x) \leq 2 \\ 0 & \text{if } a(x) > 2 \end{cases}$$

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