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Fine ceramics (advanced ceramics, advanced technical ceramics) — Weibull statistics for strength data

Céramiques techniques — Statistiques Weibull des données de résistance

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Foreword

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ISO 20501 was prepared by Technical Committee ISO/TC 206, *Fine ceramics*.

Fine ceramics (advanced ceramics, advanced technical ceramics) — Weibull statistics for strength data

1 Scope

This International Standard covers the reporting of uniaxial strength data and the estimation of probability distribution parameters for advanced ceramics which fail in a brittle fashion. The failure strength of advanced ceramics is treated as a continuous random variable. Typically, a number of test specimens with well-defined geometry are brought to failure under well-defined isothermal loading conditions. The load at which each specimen fails is recorded. The resulting failure stresses are used to obtain parameter estimates associated with the underlying population distribution.

This International Standard is restricted to the assumption that the distribution underlying the failure strengths is the two-parameter Weibull distribution with size scaling. Furthermore, this International Standard is restricted to test specimens (tensile, flexural, pressurized ring, etc.) that are primarily subjected to uniaxial stress states. Subclauses 5.4 and 5.5 outline methods of correcting for bias errors in the estimated Weibull parameters, and to calculate confidence bounds on those estimates from data sets where all failures originate from a single flaw population (i.e., a single failure mode). In samples where failures originate from multiple independent flaw populations (e.g., competing failure modes), the methods outlined in 5.4 and 5.5 for bias correction and confidence bounds are not applicable.

Measurements of the strength at failure are taken for one of two reasons: either for a comparison of the relative quality of two materials, or the prediction of the probability of failure (or alternatively the fracture strength) for a structure of interest. This International Standard permits estimates of the distribution parameters which are needed for either. In addition, this International Standard encourages the integration of mechanical property data and fractographic analysis.

2 Terms and definitions

For the purposes of this document, the following terms and definitions apply.

2.1 Defect populations

2.1.1
censored strength data
strength measurements (i.e., a sample) containing suspended observations such as that produced by multiple competing or concurrent flaw populations

NOTE Consider a sample where fractography clearly established the existence of three concurrent flaw distributions (although this discussion is applicable to a sample with any number of concurrent flaw distributions). The three concurrent flaw distributions are referred to here as distributions A, B, and C. Based on fractographic analyses, each specimen strength is assigned to a flaw distribution that initiated failure. In estimating parameters that characterize the strength distribution associated with flaw distribution A, all specimens (and not just those that failed from type-A flaws) must be incorporated in the analysis to assure efficiency and accuracy of the resulting parameter estimates. The strength of a specimen that failed by a type-B (or type-C) flaw is treated as a *right censored* observation relative to the A flaw distribution. Failure due to a type-B (or type-C) flaw restricts, or censors, the information concerning type-A flaws in a specimen by suspending the test before failure occurs by a type-A flaw [2]. The strength from the most severe type-A flaw in those specimens that failed from type-B (or type-C) flaws is higher than (and thus to the *right* of) the observed strength. However, no information is provided regarding the magnitude of that difference. Censored data analysis techniques incorporated in this International Standard utilize this incomplete information to provide efficient and relatively unbiased estimates of the distribution parameters.

2.1.2
competing failure modes
distinguishably different types of fracture initiation events that result from concurrent (competing) flaw distributions

2.1.3
compound flaw distributions
any form of multiple flaw distribution that is neither pure concurrent, nor pure exclusive

NOTE A simple example is where every specimen contains the flaw distribution A, while some fraction of the specimens also contains a second independent flaw distribution B.

2.1.4
concurrent flaw distributions
a type of multiple flaw distribution in a homogeneous material where every specimen of that material contains representative flaws from each independent flaw population

NOTE Within a given specimen, all flaw populations are then present concurrently and are competing with each other to cause failure. This term is synonymous with “competing flaw distributions”.

2.1.5
exclusive flaw distributions
a type of multiple flaw distribution created by mixing and randomizing specimens from two or more versions of a material where each version contains a different single flaw population

NOTE Thus, each specimen contains flaws exclusively from a single distribution, but the total data set reflects more than one type of strength-controlling flaw. This term is synonymous with “mixture flaw distributions”.

2.1.6
extraneous flaws
strength-controlling flaws observed in some fraction of test specimens that cannot be present in the component being designed

NOTE An example is machining flaws in ground bend specimens that will not be present in as-sintered components of the same material.

2.2 Mechanical testing

2.2.1
effective gauge section
that portion of the test specimen geometry included within the limits of integration (volume, area or edge length) of the Weibull distribution function

NOTE In tensile specimens, the integration may be restricted to the uniformly stressed central gauge section, or it may be extended to include transition and shank regions.

2.2.2
fractography
the analysis and characterization of patterns generated on the fracture surface of a test specimen

NOTE Fractography can be used to determine the nature and location of the critical fracture origin causing catastrophic failure in an advanced ceramic test specimen or component.

2.2.3
proof testing
applying a predetermined load to every test specimen (or component) in a batch or a lot over a short period of time to ascertain if the specimen fails due to a serious strength limiting defect

NOTE This procedure, when applied to all specimens in the sample, removes potentially weak specimens and modifies the statistical characteristics of the surviving samples.

2.3 Statistical terms

2.3.1
confidence interval
interval within which one would expect to find the true population parameter

NOTE Confidence intervals are functionally dependent on the type of estimator utilized and the sample size. The level of expectation is associated with a given confidence level. When confidence bounds are compared to the parameter estimate one can quantify the uncertainty associated with a point estimate of a population parameter.

2.3.2
confidence level
probability that the true population parameter falls within a specified confidence interval

2.3.3
estimator
well-defined function that is dependent on the observations in a sample

NOTE The resulting value for a given sample may be an estimate of a distribution parameter (a point estimate) associated with the underlying population. The arithmetic average of a sample is, e.g., an estimator of the distribution mean.

2.3.4
population
totality of potential observations about which inferences are made

2.3.5
population mean
the average of all potential measurements in a given population weighted by their relative frequencies in the population

2.3.6
probability density function
function $f(x)$ is a probability density function for the continuous random variable X if

$$f(x) \geqslant 0 \tag{1}$$

and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{2}$$

NOTE The probability that the random variable X assumes a value between a and b is given by

$$Pr(a < X < b) = \int_{a}^{b} f(x)\,dx \tag{3}$$

2.3.7
ranking estimator
function that estimates the probability of failure to a particular strength measurement within a ranked sample

2.3.8
sample
collection of measurements or observations taken from a specified population

2.3.9
skewness
term relating to the asymmetry of a probability density function

NOTE The distribution of failure strength for advanced ceramics is not symmetric with respect to the maximum value of the distribution function but has one tail longer than the other.

2.3.10
statistical bias
inherent to most estimates, this is a type of consistent numerical offset in an estimate relative to the true underlying value

NOTE The magnitude of the bias error typically decreases as the sample size increases.

2.3.11
unbiased estimator
estimator that has been corrected for statistical bias error

2.4 Weibull distributions

2.4.1
Weibull distribution
continuous random variable X has a two-parameter Weibull distribution if the probability density function is given by

$$f(x) = \left(\frac{m}{\beta}\right)\left(\frac{x}{\beta}\right)^{m-1} \exp\left[-\left(\frac{x}{\beta}\right)^{m}\right] \text{ when } x > 0 \tag{4}$$

or

$$f(x) = 0 \text{ when } x \leqslant 0 \tag{5}$$

and the cumulative distribution function is given by

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{m}\right] \text{ when } x > 0 \tag{6}$$

or

$$F(x) = 0 \text{ when } x \leqslant 0 \tag{7}$$

where

m is the Weibull modulus (or the shape parameter) (> 0);

β is the Weibull scale parameter (> 0)

NOTE 1 The random variable representing uniaxial tensile strength of an advanced ceramic will assume only positive values, and the distribution is asymmetrical about the mean. These characteristics rule out the use of the normal distribution (as well as others) and point to the use of the Weibull and similar skewed distributions. If the random variable representing uniaxial tensile strength of an advanced ceramic is characterized by Equations 4 to 7, then the probability that this advanced ceramic will fail under an applied uniaxial tensile stress σ is given by the cumulative distribution function

$$P_{\mathrm{f}} = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_{\theta}}\right)^{m}\right] \text{ when } \sigma > 0 \tag{8}$$

$$P_{\mathrm{f}} = 0 \text{ when } \sigma \leqslant 0 \tag{9}$$

where

P_{f} is the probability of failure;

σ_{θ} is the Weibull characteristic strength.

NOTE 2 The Weibull characteristic strength is dependent on the uniaxial test specimen (tensile, flexural, or pressurized ring) and will change with specimen geometry. In addition, the Weibull characteristic strength has units of stress, and should be reported using units of MPa or GPa.

NOTE 3 An alternative expression for the probability of failure is given by

$$P_{\mathrm{f}} = 1 - \exp\left[-\int_V \left(\frac{\sigma}{\sigma_0}\right)^m dV\right] \text{ when } \sigma > 0 \quad (10)$$

$$P_{\mathrm{f}} = 0 \text{ when } \sigma \leqslant 0 \quad (11)$$

The integration in the exponential is performed over all tensile regions of the specimen volume if the strength-controlling flaws are randomly distributed through the volume of the material, or over all tensile regions of the specimen area if flaws are restricted to the specimen surface. The integration is sometimes carried out over an effective gauge section instead of over the total volume or area. In Equation 10, σ_0 is the Weibull material scale parameter and can be described as the Weibull characteristic strength of a specimen with unit volume or area loaded in uniform uniaxial tension. The Weibull material scale parameter has units of stress·(volume)$^{1/m}$, and should be reported using units of MPa·m$^{3/m}$ or GPa·m$^{3/m}$ if the strength-controlling flaws are distributed through the volume of the material. If the strength-controlling flaws are restricted to the surface of the specimens in a sample, then the Weibull material scale parameter should be reported using units of MPa·m$^{2/m}$ or GPa·m$^{2/m}$. For a given specimen geometry, Equations 8 and 10 can be combined, to yield an expression relating σ_0 and σ_θ. Further discussion related to this issue can be found in Annex A.

3 Symbols

A specimen area

b gauge section dimension, base of bend test specimen

d gauge section dimension, depth of bend test specimen

$f(x)$ probability density function

$F(x)$ cumulative distribution function

L likelihood function

L_{i} length of the inner load span for a bend test specimen

L_{o} length of the outer load span for a bend test specimen

m Weibull modulus

$\hat{m}$ estimate of the Weibull modulus

$\hat{m}_{\mathrm{U}}$ unbiased estimate of the Weibull modulus

N number of specimens in a sample

P_{f} probability of failure

r number of specimens that failed from the flaw population for which the Weibull estimators are being calculated

t intermediate quantity defined by Equation 15, used in calculation of confidence bounds

V specimen volume

x realization of a random variable X

X random variable

β Weibull scale parameter

σ uniaxial tensile stress

$\hat{\sigma}$ estimate of mean strength

σ_j maximum stress in the j th test specimen at failure

σ_0 Weibull material scale parameter (strength relative to unit size) defined in Equation 10

$\hat{\sigma}_0$ estimate of the Weibull material scale parameter

σ_θ Weibull characteristic strength (associated with a test specimen) defined in Equation 8

$\hat{\sigma}_\theta$ estimate of the Weibull characteristic strength

4 Significance and use

4.1 This International Standard enables the experimentalist to estimate Weibull distribution parameters from failure data. These parameters permit a description of the statistical nature of fracture of fine ceramic materials for a variety of purposes, particularly as a measure of reliability as it relates to strength data utilized for mechanical design purposes. The observed strength values are dependent on specimen size and geometry. Parameter estimates can be computed for a given specimen geometry $(\hat{m}, \hat{\sigma}_\theta)$, but it is suggested that the parameter estimates be transformed and reported as material-specific parameters $(\hat{m}, \hat{\sigma}_0)$. In addition, different flaw distributions (e.g., failures due to inclusions or machining damage) may be observed, and each will have its own strength distribution parameters. The procedure for transforming parameter estimates for typical specimen geometries and flaw distributions is outlined in Annex A.

4.2 This International Standard provides two approaches, Method A and Method B, which are appropriate for different purposes.

Method A provides a simple analysis for circumstances in which the nature of strength-defining flaws is either known or assumed to be from a single population. Fractography to identify and group test items with given flaw types is thus not required. This method is suitable for use for simple material screening.

Method B provides an analysis for the general case in which competing flaw populations exist. This method is appropriate for final component design and analysis. The method requires that fractography be undertaken to identify the nature of strength-limiting flaws and assign failure data to given flaw population types.

4.3 In method A, a strength data set can be analysed and values of the Weibull modulus and characteristic strength $(\hat{m}, \hat{\sigma}_\theta)$ are produced, together with confidence bounds on these parameters. If necessary the estimate of the mean strength can be computed. Finally, a graphical representation of the failure data along with a test report can be prepared. It should be noted that the confidence bounds are frequently widely spaced, which indicates that the results of the analysis should not be used to extrapolate far beyond the existing bounds of probability of failure.

4.4 In method B, begin by performing a fractographic examination of each failed specimen in order to characterize fracture origins. Screen the data associated with each flaw distribution for outliers. If all failures originate from a single flaw distribution compute an unbiased estimate of the Weibull modulus, and compute confidence bounds for both the estimated Weibull modulus and the estimated Weibull characteristic strength. If the failures originate from more than one flaw type, separate the data sets associated with each flaw type, and subject these individually to the censored analysis. Finally, prepare a graphical representation of the failure data along with a test report. When using the results of the analysis for design purposes it should be noted that there is an implicit assumption that the flaw populations in the strength test pieces and the components are of the same types.

5 Method A: maximum likelihood parameter estimators for single flaw populations

5.1 General

This International Standard outlines the application of parameter estimation methods based on the maximum likelihood technique. This technique has certain advantages. The parameter estimates obtained using the maximum likelihood technique are unique (for a two-parameter Weibull distribution), and as the size of the sample increases, the estimates statistically approach the true values of the population more efficiently than other parameter estimation techniques.

5.2 Censored data

The application of the techniques presented in this International Standard can be complicated by the presence of test specimens that fail from extraneous flaws, fractures that originate outside the effective gauge section, and unidentified fracture origins. If these complications arise, the strength data from these specimens should generally not be discarded. Strength data from specimens with fracture origins outside the effective gauge section [3] and from specimens with fractures that originate from extraneous flaws should be censored, and the maximum likelihood methods presented later in Method B (Clause 6) of this International Standard are applicable. It is imperative that the number of unidentified fracture origins, and how they were classified, be stated in the test report. A discussion of the appropriateness of each option can be found in 6.2.2.

5.3 Likelihood functions

The likelihood function for the two-parameter Weibull distribution of a sample with a single flaw population [4] is defined by the expression:

$$L = \prod_{i=1}^{N} \left(\frac{\hat{m}}{\hat{\sigma}_\theta} \right) \left(\frac{\sigma_i}{\hat{\sigma}_\theta} \right)^{\hat{m}-1} \exp\left[-\left(\frac{\sigma_i}{\hat{\sigma}_\theta} \right)^{\hat{m}} \right] \quad (12)$$

NOTE σ_i is the maximum stress in the i th test specimen at failure and N is the number of test specimens in the sample being analysed. The parameter estimates (the Weibull modulus, $\hat{m}$, and the characteristic strength, $\hat{\sigma}_\theta$) are determined by taking the partial derivatives of the logarithm of the likelihood function with respect to $\hat{m}$ and $\hat{\sigma}_\theta$ and equating the resulting expressions to zero.

The system of equations obtained by differentiating the log likelihood function for a sample with a single flaw population [5] is given by

$$\frac{\sum_{i=1}^{N} (\sigma_i)^{\hat{m}} \ln(\sigma_i)}{\sum_{i=1}^{N} (\sigma_i)^{\hat{m}}} - \frac{1}{N} \sum_{i=1}^{N} \ln(\sigma_i) - \frac{1}{\hat{m}} = 0 \quad (13)$$

and

$$\hat{\sigma}_\theta = \left[\left(\sum_{i=1}^{N} (\sigma_i)^{\hat{m}} \right) \frac{1}{N} \right]^{1/\hat{m}} \tag{14}$$

Equation 13 is solved first for $\hat{m}$. Subsequently $\hat{\sigma}_\theta$ is computed from Equation 14. Obtaining a closed form solution of Equation 13 for $\hat{m}$ is not possible. This expression must be solved numerically.

Since the characteristic strength also reflects specimen geometry and stress gradients, this International Standard suggests reporting the estimated Weibull material scale parameter, $\hat{\sigma}_0$. Expressions that relate $\hat{\sigma}_\theta$ to the Weibull material scale parameter σ_0 for typical specimen geometries are given in Annex A.

5.4 Bias correction

5.4.1 The procedures described herein, to correct for statistical bias errors and to compute confidence bounds, are appropriate only for data sets where all failures originate from a single population (i.e., an uncensored sample). Procedures for bias correction and confidence bounds in the presence of multiple active flaw populations are not currently well developed. The statistical bias associated with the estimator $\hat{\sigma}_\theta$ is minimal (< 0,3 % for 20 test specimens, as opposed to ≈ 7 % bias for $\hat{m}$ with the same number of specimens). Therefore, this International Standard allows the assumption that $\hat{\sigma}_\theta$ is an unbiased estimator of the true population parameter. The parameter estimate of the Weibull modulus, $\hat{m}$, generally exhibits statistical bias. The amount of statistical bias depends on the number of specimens in the sample. An unbiased estimate of $\hat{m}$ shall be obtained by multiplying $\hat{m}$ by unbiasing factors [6]. This procedure is discussed in 5.4.2. Statistical bias associated with the maximum likelihood estimators presented in this International Standard can be reduced by increasing the sample size.

5.4.2 An unbiased estimator produces nearly zero statistical bias between the value of the true parameter and the point estimate. The amount of deviation can be quantified either as a percent difference or with unbiasing factors. In keeping with the accepted practice in the open literature, this International Standard quantifies statistical bias through the use of unbiasing factors, denoted here as UF. Depending on the number of specimens in a given sample, the point estimate of the Weibull modulus, $\hat{m}$, may exhibit significant statistical bias. An unbiased estimate of the Weibull modulus (denoted as $\hat{m}_U$) is obtained by multiplying the biased estimate with an appropriate unbiasing factor. Unbiasing factors for $\hat{m}$ are listed in Table 1. An example in Annex B demonstrates the use of Table 1 in correcting a biased estimate of the Weibull modulus.

Table 1 — Unbiasing factor for the maximum likelihood estimate of the Weibull modulus

Number of specimens, N	Unbiasing factor, UF	Number of specimens, N	Unbiasing factor, UF
5	0,700	42	0,968
6	0,752	44	0,970
7	0,792	46	0,971
8	0,820	48	0,972
9	0,842	50	0,973
10	0,859	52	0,974
11	0,872	54	0,975
12	0,883	56	0,976
13	0,893	58	0,977
14	0,901	60	0,978
15	0,908	62	0,979
16	0,914	64	0,980
18	0,923	66	0,980
20	0,931	68	0,981
22	0,938	70	0,981
24	0,943	72	0,982
26	0,947	74	0,982
28	0,951	76	0,983
30	0,955	78	0,983
32	0,958	80	0,984
34	0,960	85	0,985
36	0,962	90	0,986
38	0,964	100	0,987
40	0,966	120	0,990

5.5 Confidence intervals

5.5.1 Confidence bounds quantify the uncertainty associated with a point estimate of a population parameter. The size of the confidence bounds for maximum likelihood estimates of both Weibull parameters will diminish with increasing sample size. The values used to construct confidence bounds are based on percentile distributions obtained by Monte Carlo simulation; e.g., the 90 % confidence bound on the Weibull modulus is obtained from the 5 and 95 percentile distributions of the ratio of $\hat{m}$ to the true population value m. For a point estimate of the Weibull modulus, the normalized values ($\hat{m}/m$) necessary to construct the 90 % confidence bounds are listed in Table 2. The example in Annex B demonstrates the use of Table 2 in constructing the upper and lower bounds. Note that the statistically biased estimate of the Weibull modulus shall be used here. Again, this procedure is not appropriate for censored statistics.

5.5.2 Confidence bounds can be constructed for the estimated Weibull characteristic strength. However, the percentile distributions needed to construct the bounds do not involve the same normalized ratios or the same tables as those used for the Weibull modulus. Define the function:

$$t = \hat{m} \ln\left(\hat{\sigma}_\theta / \sigma_\theta \right) \quad (15)$$

The 90 % confidence bound on the characteristic strength is obtained from the 5 and 95 percentile distributions of t. For the point estimate of the characteristic strength, these percentile distributions are listed in Table 3. An example in Annex B demonstrates the use of Table 3 in constructing upper and lower bounds on $\hat{\sigma}_\theta$. Note that the biased estimate of the Weibull modulus shall be used here. Again, this procedure is not appropriate for censored statistics. Note that Equation 15 is not applicable for developing confidence bounds on $\hat{\sigma}_0$, therefore the confidence bounds on $\hat{\sigma}_\theta$ should not be converted through the use of Equations 8 and 10.

Table 2 — Normalized upper and lower bounds on the maximum likelihood estimate of the Weibull modulus — 90 % confidence interval

Number of specimens, N	$q_{0,05}$	$q_{0,95}$	Number of specimens, N	$q_{0,05}$	$q_{0,95}$
5	0,683	2,779	42	0,842	1,265
6	0,697	2,436	44	0,845	1,256
7	0,709	2,183	46	0,847	1,249
8	0,720	2,015	48	0,850	1,242
9	0,729	1,896	50	0,852	1,235
10	0,738	1,807	52	0,854	1,229
11	0,745	1,738	54	0,857	1,224
12	0,752	1,682	56	0,859	1,218
13	0,759	1,636	58	0,861	1,213
14	0,764	1,597	60	0,863	1,208
15	0,770	1,564	62	0,864	1,204
16	0,775	1,535	64	0,866	1,200
17	0,779	1,510	66	0,868	1,196
18	0,784	1,487	68	0,869	1,192
19	0,788	1,467	70	0,871	1,188
20	0,791	1,449	72	0,872	1,185
22	0,798	1,418	74	0,874	1,182
24	0,805	1,392	76	0,875	1,179
26	0,810	1,370	78	0,876	1,176
28	0,815	1,351	80	0,878	1,173
30	0,820	1,334	85	0,881	1,166
32	0,824	1,319	90	0,883	1,160
34	0,828	1,306	95	0,886	1,155
36	0,832	1,294	100	0,888	1,150
38	0,835	1,283	110	0,893	1,141
40	0,839	1,273	120	0,897	1,133

Table 3 — Normalized upper and lower bounds on the function t — 90 % confidence interval

Number of specimens, N	$t_{0,05}$	$t_{0,95}$	Number of specimens, N	$t_{0,05}$	$t_{0,95}$
5	− 1,247	1,107	42	− 0,280	0,278
6	− 1,007	0,939	44	− 0,273	0,271
7	− 0,874	0,829	46	− 0,266	0,264
8	− 0,784	0,751	48	− 0,260	0,258
9	− 0,717	0,691	50	− 0,254	0,253
10	− 0,665	0,644	52	− 0,249	0,247
11	− 0,622	0,605	54	− 0,244	0,243
12	− 0,587	0,572	56	− 0,239	0,238
13	− 0,557	0,544	58	− 0,234	0,233
14	− 0,532	0,520	60	− 0,230	0,229
15	− 0,509	0,499	62	− 0,226	0,225
16	− 0,489	0,480	64	− 0,222	0,221
17	− 0,471	0,463	66	− 0,218	0,218
18	− 0,455	0,447	68	− 0,215	0,214
19	− 0,441	0,433	70	− 0,211	0,211
20	− 0,428	0,421	72	− 0,208	0,208
22	− 0,404	0,398	74	− 0,205	0,205
24	− 0,384	0,379	76	− 0,202	0,202
26	− 0,367	0,362	78	− 0,199	0,199
28	− 0,352	0,347	80	− 0,197	0,197
30	− 0,338	0,334	85	− 0,190	0,190
32	− 0,326	0,323	90	− 0,184	0,185
34	− 0,315	0,312	95	− 0,179	0,179
36	− 0,305	0,302	100	− 0,174	0,175
38	− 0,296	0,293	110	− 0,165	0,166
40	− 0,288	0,285	120	− 0,158	0,159

6 Method B: maximum likelihood parameter estimators for competing flaw populations

6.1 General

This International Standard outlines the application of parameter estimation methods based on the maximum likelihood technique. This technique has certain advantages, especially when parameters must be determined from censored failure populations. When a sample of test specimens yields two or more distinct flaw distributions, the sample is said to contain censored data, and the associated methods for censored data must be used. The methods described in this International Standard include censoring techniques that apply to multiple concurrent flaw distributions. However, the techniques for parameter estimation presented in this International Standard are not directly applicable to data sets that contain exclusive or compound multiple flaw distributions [7].

The estimation techniques for censored data presented herein require positive confirmation of multiple flaw distributions, which necessitates fractographic examination in order to characterize the fracture origin in each specimen. Multiple flaw distributions may be further evidenced by deviation from the linearity of the data from a single Weibull distribution. However, since there are many exceptions, observations of approximately linear behaviour should not be considered sufficient reason to conclude that only a single flaw distribution is active.

For data sets with multiple active flaw distributions where one flaw distribution (identified by fractographic analysis) occurs in a small number of specimens, it is sufficient to report the existence of this flaw distribution (and the number of occurrences), but it is not necessary to estimate Weibull parameters. Estimates of the Weibull parameters for this flaw distribution would be potentially biased with wide confidence bounds (neither of which could be quantified). However, special note should be made in the report if the occurrences of this flaw distribution take place in the upper or lower tail of the sample strength distribution.

6.2 Censored data

6.2.1 The application of the censoring techniques presented in this International Standard can be complicated by the presence of test specimens that fail from extraneous flaws, fractures that originate outside the effective gauge section, and unidentified fracture origins. If these complications arise, the strength data from these specimens should generally not be discarded. Strength data from specimens with fracture origins outside the effective gauge section [3] as well as from specimens with fractures that originate from extraneous flaws should be censored, and the maximum likelihood methods presented in this International Standard are applicable.

6.2.2 This International Standard recognizes four options the experimentalist can pursue when unidentified fracture origins are encountered during fractographic examinations. Specimens with unidentified fracture origins can be:

a) assigned a previously identified flaw distribution using inferences based on all available fractographic information;

b) assigned the same flaw distribution as that of the specimen closest in strength;

c) assigned a new and as yet unspecified flaw distribution;

d) be removed from the sample.

6.2.3 It is imperative that the number of unidentified fracture origins, and how they were classified, be stated in the test report. A discussion of the appropriateness of each option appears in Annex C. If the strength data and the resulting parameter estimates are used for component design, the engineer must consult with the fractographer before and after performing the fractographic examination. Considerable judgement may be needed to identify the correct option. Whenever partial fractographic information is available option a) is strongly recommended, especially if the data are used for component design. Conversely, option d) is not recommended by this International Standard unless there is overwhelming justification.

6.3 Likelihood functions

The likelihood function for the two-parameter Weibull distribution of a censored sample is defined by equation [4]:

$$L = \left\{ \prod_{i=1}^{r} \left(\frac{\hat{m}}{\hat{\sigma}_\theta} \right) \left(\frac{\sigma_i}{\hat{\sigma}_\theta} \right)^{\hat{m}-1} \exp\left[-\left(\frac{\sigma_i}{\hat{\sigma}_\theta} \right)^{\hat{m}} \right] \right\} \left\{ \prod_{j=r+1}^{N} \exp\left[-\left(\frac{\sigma_j}{\hat{\sigma}_\theta} \right)^{\hat{m}} \right] \right\} \quad (16)$$

This expression is applied to a sample where two or more active concurrent flaw distributions have been identified from fractographic inspection. For the purpose of the discussion here, the different distributions are identified as flaw types A, B, C, etc. When Equation 16 is used to estimate the parameters associated with the "A" flaw distribution, then r is the number of specimens where type-A flaws were found at the fracture origin, and i is the associated index in the first product. The second product is carried out for all other specimens not failing from type-A flaws (i.e., type-B flaws, type-C flaws, etc.). Therefore the product is carried out from ($j = r + 1$) to N (the total number of specimens) where j is the index in the second product. Accordingly, σ_i and σ_j are the maximum stress in the ith and jth test specimen at failure. The parameter estimates (the Weibull modulus, $\hat{m}$, and the characteristic strength, σ_θ) are determined by taking the partial derivatives of the logarithm of the likelihood function with respect to $\hat{m}$ and $\hat{\sigma}_\theta$ and equating the resulting expressions to zero. Note that $\hat{\sigma}_\theta$ is a function of specimen geometry and the estimate of the Weibull modulus $\hat{m}$. Expressions that relate $\hat{\sigma}_\theta$ to the Weibull material scale parameter σ_0 for typical specimen geometries are given in Annex A.

The system of equations obtained by differentiating the log likelihood function for a censored sample [5] is given by:

$$\frac{\sum_{i=1}^{N}(\sigma_i)^{\hat{m}}\ln(\sigma_i)}{\sum_{i=1}^{N}(\sigma_i)^{\hat{m}}}-\frac{1}{r}\sum_{i=1}^{r}\ln(\sigma_i)-\frac{1}{\hat{m}}=0 \tag{17}$$

and

$$\hat{\sigma}_\theta=\left[\left(\sum_{i=1}^{N}(\sigma_i)^{\hat{m}}\right)\frac{1}{r}\right]^{1/\hat{m}} \tag{18}$$

where r is the number of failed specimens from a particular group of a censored sample.

Equation 17 is solved first for $\hat{m}$. Subsequently $\hat{\sigma}_\theta$ is computed from Equation 18. Obtaining a closed form solution of Equation 17 for $\hat{m}$ is not possible. This expression must be solved numerically.

7 Procedure

7.1 Outlying observations

Before computing the parameter estimates, the data should be screened for outlying observations (outliers). An outlying observation is one that deviates significantly from other observations in the sample. It should be understood that an apparent outlying observation may be an extreme manifestation of the variability of the strength of an advanced ceramic. If this is the case, the data point should be retained and treated as any other observation in the failure sample. However, the outlying observation may be the result of a gross deviation from prescribed experimental procedure or an error in calculating or recording the numerical value of the data point in question. When the experimentalist is clearly aware that a gross deviation from the prescribed experimental procedure has occurred, the outlying observation may be discarded, unless the observation can be corrected in a rational manner.

7.2 Fractography

7.2.1 Fractographic examination of each failed specimen is highly recommended in order to characterize the fracture origins. The strength of advanced ceramics is often limited by discrete fracture origins which may be intrinsic or extrinsic to the material. Porosity, agglomerates, inclusions, and atypical large grains are considered intrinsic fracture origins. Extrinsic fracture origins are typically on the surface of the specimen and are the result of contact stresses, impact events or adverse environment. When the means are available to the experimentalist, fractographic methods should be used to locate, identify, and classify the strength-limiting fracture origin causing catastrophic failure in an advanced ceramic test specimen. Moreover, for the purpose of parameter estimation, each classification of fracture origin must be identified as a surface fracture origin or a volume fracture origin in order to use the expressions given in Annex A. Thus, there may exist several classifications of fracture origins within the volume (or surface area) of the test specimens in a sample. It should be clearly indicated on the test report if a fractographic analysis is not performed.

7.2.2 Perform a fractographic analysis and label each datum with a symbol identifying the type of fracture origin. This may be a word, an abbreviation, or a different symbol for each type of fracture origin.

7.3 Graphical representation

7.3.1 An objective of this International Standard is the consistent representation of strength data. To this end, the following procedure is the recommended graphical representation of strength data. Begin by ranking the strength data obtained from laboratory testing in ascending order, and assign to each a ranked probability of failure P_f according to the estimator

$$P_f(\sigma_i) = \frac{i - 0{,}5}{N} \quad (19a)$$

or

$$P_f(\sigma_i) = \frac{i - 0{,}3}{N + 0{,}4} \quad (19b)$$

where

N is the number of specimens;

i is the ith datum.

Compute the natural logarithm of the ith failure stress, and the natural logarithm of the natural logarithm of $[1/(1 - P_f)]$ (i.e., the double logarithm of the quantity in square brackets), where P_f is associated with the ith failure stress.

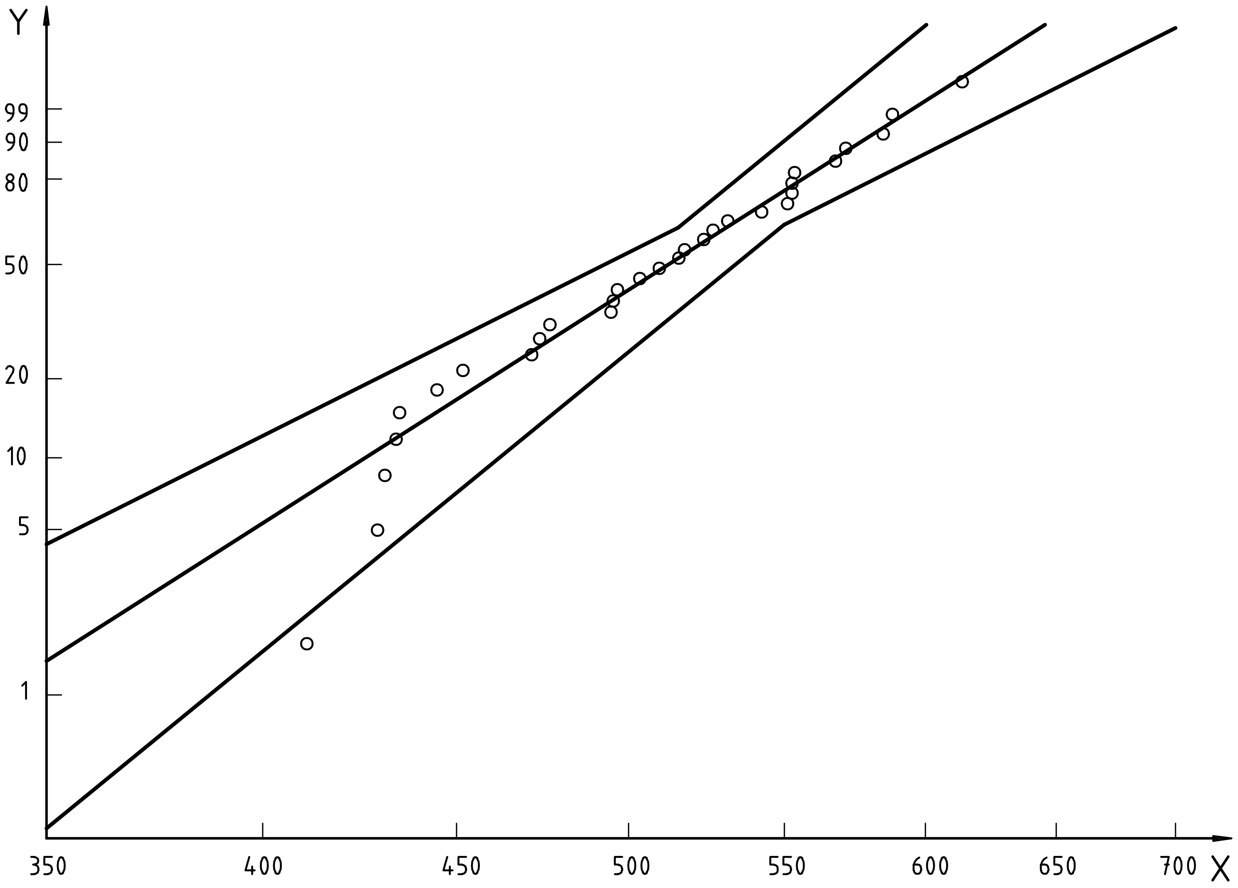
7.3.2 Create a graph representing the data as shown in Figure 1. Plot $\ln\{\ln[1/(1 - P_f)]\}$ as the ordinate, and $\ln(s)$ as the abscissa. A typical ordinate scale assumes values from + 2 to − 6. This approximately corresponds to a range in probability of failure from 0,25 % to 99,9 %. The ordinate axis must be labelled as probability of failure P_f, as depicted in Figure 1. Similarly, the abscissa must be labelled as failure stress (flexural, tensile, etc.), preferably using units of MPa or GPa.

7.3.3 Included on the plot should be a line whose position is fixed by the estimates of the Weibull parameters. The line is defined by the following mathematical expression:

$$P_f = 1 - \exp\left[-\left(\frac{\sigma}{\hat{\sigma}_\theta}\right)^{\hat{m}}\right] \quad (20)$$

The slope of the line, which is the estimate of the Weibull modulus $\hat{m}$, should be identified, as shown in Figure 1. The estimate of the characteristic strength $\hat{\sigma}_\theta$ should also be identified. This corresponds to a P_f of 63,2 %, or a value of zero for $\ln\{\ln[1/(1 - P_f)]\}$.

7.3.4 This International Standard does not provide a definitive criterion in order to judge the relative fit of the individual data points to a linear two-parameter Weibull curve estimated from the data. Theoretical bounds on the reliability curve are complex and outside the scope of this International Standard. Confidence bounds on the estimate of the Weibull modulus and the Weibull characteristic strength can be used to construct confidence bands in a Weibull plot (see Figure 1). The bands to the left of the estimated two-parameter Weibull curve are constructed using the lower confidence bound on the Weibull characteristic strength and the upper confidence bound on the Weibull modulus for probability of failures above 63,2 %. Probability of failures below 63,2 % correspond to the lower confidence bound on the Weibull modulus. The bands to the right of the estimated two-parameter Weibull curve are constructed using the upper confidence bound on the Weibull characteristic strength along with the lower confidence bound on the Weibull modulus for probability of failures above 63,2 %. Probabilities of failure below 63,2 % correspond to the upper confidence bound on the Weibull modulus.



Failure stresses, MPa	
411	516
429	518
431	524
434	527
435	532
445	543
452	552
472	553
474	553
477	554
495	568
496	572
497	585
504	588
510	614

Key

X Failure stress

Y Probability of failure, %

NOTE Estimated $m = 10{,}24$; $\sigma_\theta = 532$ MPa.

Figure 1 — Weibull Plot

8 Test report

The test report shall contain the following information:

a) type of material characterized;

b) test procedure (preferably designating an appropriate standard);

c) number of failed specimens;

d) flaw type;

e) maximum likelihood estimates of the Weibull parameters;

f) unbiasing factor;

g) information that allows the construction of 90 % confidence bounds (accompanying the graph in order to provide a complete representation of the failure data).

Insert a column on the graph (in any convenient location), or alternatively provide a separate table that identifies the individual strength values in ascending order. This will permit other users to perform alternative analyses. In addition, the experimentalist should include a separate sketch of the specimen geometry that includes all pertinent dimensions. An estimate of mean strength can also be depicted in the graph. The estimate of mean strength is calculated by using the arithmetic mean as the estimator:

$$\hat{\mu} = \left(\sum_{i=1}^{N} \sigma_i \right) \left(\frac{1}{N} \right) \tag{21}$$

NOTE This estimate of the mean strength is not appropriate for samples with multiple failure populations.

Annex A
(informative)

Converting to material-specific strength distribution parameters

A.1 The following equation defines the relationship between the parameters for tensile specimens:

$$\left(\hat{\sigma}_0\right)_V = \left(V\right)^{1/\left(\hat{m}\right)_V}\left(\hat{\sigma}_\theta\right)_V \tag{A.1}$$

where V is the volume of the uniform gauge section of the tensile specimen, and the fracture origins are spatially distributed strictly within this volume.

The gauge section of a tensile specimen is defined herein as the central region of the test specimen with the smallest constant cross-sectional area. However, the experimentalist may include transition regions and the shank regions of the specimen if the volume (or area) integration defined by Equation 10 is analysed properly. This procedure is discussed in Clause A.3. For a tensile specimen in which fracture origins are spatially distributed strictly at the surface of the specimens tested,

$$\left(\hat{\sigma}_0\right)_A = \left(A\right)^{1/\left(\hat{m}\right)_A}\left(\hat{\sigma}_\theta\right)_A \tag{A.2}$$

where A is the surface area of the uniform gauge section.

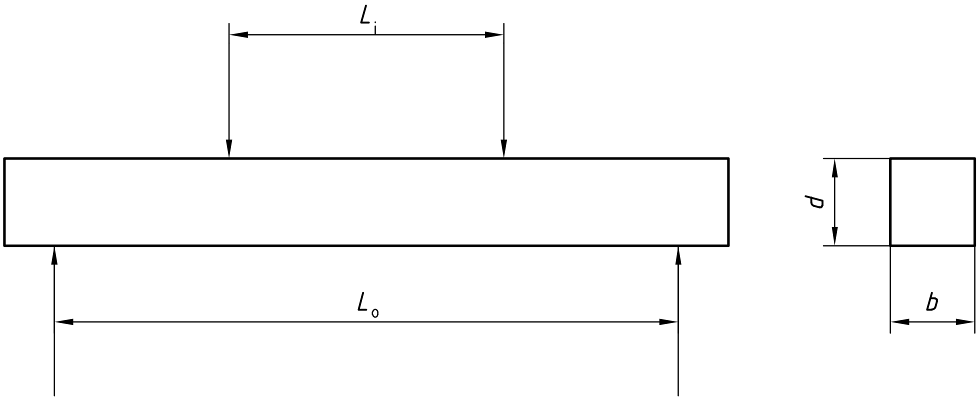


Figure A.1 — Typical bend specimen geometry

A.2 For flexural specimen geometries, the relationships become more complex [8]. The following relationship is based on the geometry of the flexural specimen shown in Figure A.1. For fracture origins spatially distributed strictly within both the volume of a flexural specimen and the outer load span:

$$\left(\hat{\sigma}_0\right)_V = \left(\hat{\sigma}_\theta\right)_V \left\{\frac{V\left[\left(\frac{L_\mathrm{i}}{L_\mathrm{o}}\right)\left(\hat{m}\right)_V + 1\right]}{2\left[\left(\hat{m}\right)_V + 1\right]^2}\right\}^{\frac{1}{\left(\hat{m}\right)_V}} \tag{A.3}$$

where

V is the volume of the gauge section defined by the expression

$$V = bdL_o \tag{A.4}$$

(b and d are dimensions identified in Figure A.1)

L_i is the length of the inner load span;

L_o is the length of the outer load span.

For fracture origins spatially distributed strictly at the surface of a flexural specimen and within the outer load span:

$$(\hat{\sigma}_0)_A = (\hat{\sigma}_\theta)_A \left\{ L_o \left[\frac{d}{\left[(\hat{m})_A + 1\right]} + b \right] \left[\left(\frac{L_i}{L_o} \right) (\hat{m})_A + 1 \right] \left[\frac{1}{(\hat{m})_A + 1} \right] \right\}^{\frac{1}{(\hat{m})_A}} \tag{A.5}$$

A.3 Test specimens other than tensile and flexural specimens may be utilized. Relationships between the estimate of the Weibull characteristic strength and the Weibull material scale parameter for any specimen configuration can be derived by equating the expressions defined by Equations 8 and 10 with the modifications that follow. Begin by replacing σ (an applied uniaxial tensile stress) in Equation 8 with σ_{max}, which is defined as the maximum tensile stress within the test specimen of interest, then:

$$P_f = 1 - \exp\left[-\left(\frac{\sigma_{max}}{\sigma_\theta} \right)^m \right] \tag{A.6}$$

Also perform the integration given in Equation 10 such that:

$$P_f = 1 - \exp\left[-kV \left(\frac{\sigma_{max}}{\sigma_0} \right)^m \right] \tag{A.7}$$

where k is a dimensionless constant that accounts for specimen geometry and stress gradients.

Note that, in general, k is a function of the estimated Weibull modulus $\hat{m}$, and is always less than or equal to unity. The product kV is often referred to as the effective volume (with the designation V_E). The effective volume can be interpreted as the size of an equivalent uniaxial tensile specimen that has the same risk of rupture as the test specimen or component. As the term implies, the product represents the volume of material subject to a uniform uniaxial tensile stress [9]. Setting Equations A.6 and A.7 equal to one another yields the following expression:

$$(\hat{\sigma}_0)_V = (kV)^{1/(\hat{m})_V} (\hat{\sigma}_\theta)_V \tag{A.8}$$

Thus for an arbitrary test specimen, the experimentalist evaluates the integral identified in Equation 10 for the effective volume, kV, and utilizes Equation A.8 to obtain the estimated Weibull material scale parameter $\hat{\sigma}_0$. A similar procedure can be adopted when fracture origins are spatially distributed at the surface of the test specimen.

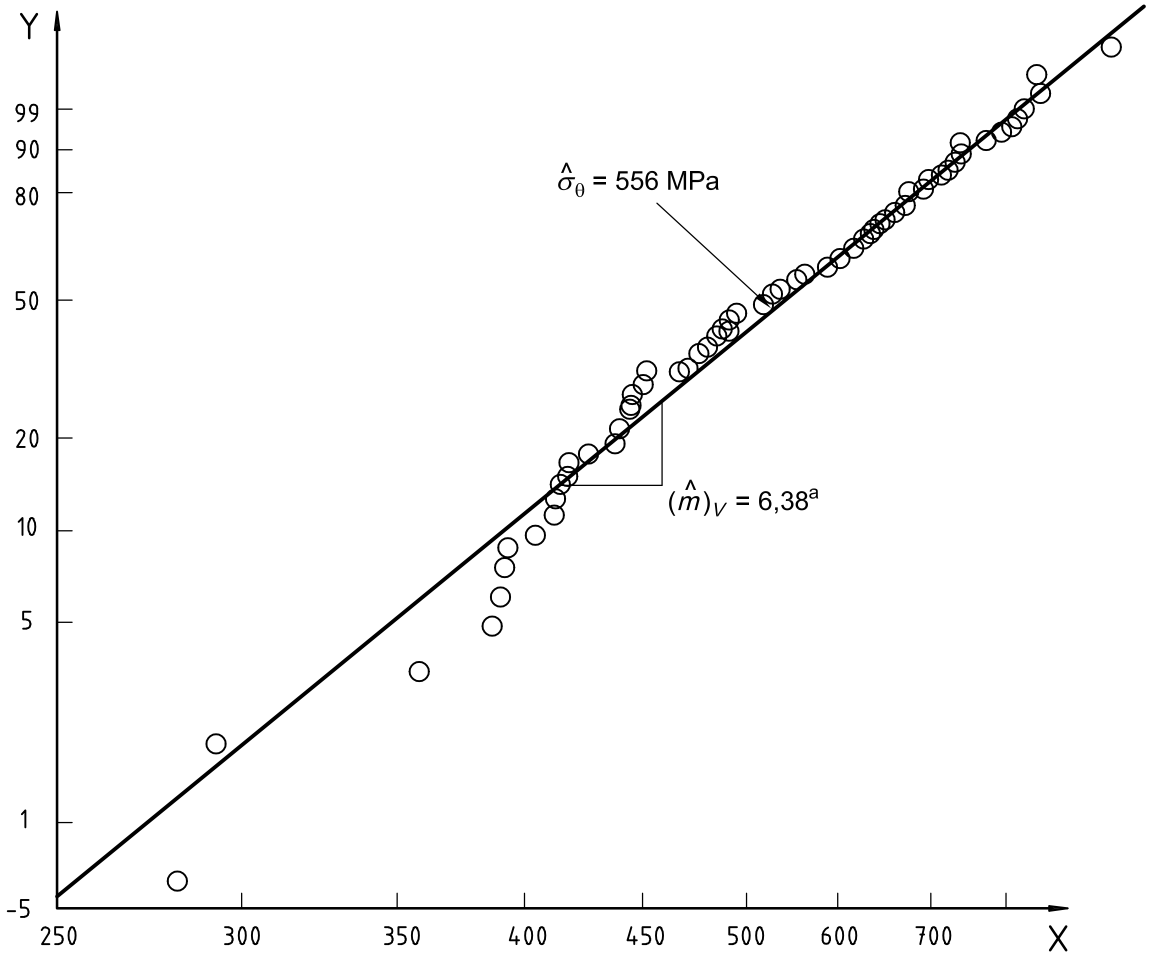
Annex B
(informative)

Illustrative examples

B.1 For the first example, consider the failure data in Table B.1. The data represent four-point (1/4 point) flexural specimens fabricated from HIP'ed (hot isostatically pressed) silicon carbide [11]. The solution of Equation 13 requires an iterative numerical scheme and for this data set yields a biased parameter estimate of $\hat{m} = 6{,}48$. Subsequent solution of Equation 14 yields a value of $\hat{\sigma}_\theta = 556$ MPa. These values for the Weibull parameters were generated by assuming a unimodal failure sample with no censoring (i.e., $r = N$). Figure B.1 depicts the individual failure data and a curve based on the estimated values of the parameters. Next, assuming that the failure origins were surface-distributed and then inserting the estimated value of $\hat{m}$ and $\hat{\sigma}_\theta$ in Equation A.5 along with the specimen geometry (i.e., $L_o = 40$ mm, $L_i = 20$ mm, $d = 3{,}5$ mm, and $b = 4{,}5$ mm) yields $(\hat{\sigma}_0)_A = 360$ MPa·$(m)^{0{,}309}$. Note that $(\hat{\sigma}_0)_A$ has units of stress·(area) $^{1/\hat{m}}$; thus, 0,309 = (2/6,48). Alternatively, if one were to assume that the failure origins were volume distributed, then the solution of Equation A.3 yields $(\hat{\sigma}_0)_V = 37{,}0$ MPa·$(m)^{0{,}463}$. Note that $(\hat{\sigma}_0)_V$ has units of stress·(volume) $^{1/\hat{m}}$; thus, 0,463 = (3/6,48). The different values obtained from assuming surface and volume fracture origins underscore the necessity of conducting a fractographic analysis.

Table B.1 — Unimodal failure stress data for HIP'ed (hot isostatically pressed) silicon carbide — Example 1

Specimen number	**Strength, σ_f** MPa	**Specimen number**	**Strength, σ_f** MPa
1	281	41	516
2	291	42	520
3	358	43	528
4	385	44	531
5	389	45	531
6	391	46	546
7	392	47	549
8	403	48	553
9	412	49	560
10	413	50	562
11	414	51	563
12	418	52	566
13	418	53	566
14	427	54	570
15	438	55	573
16	440	56	575
17	441	57	576
18	442	58	580
19	444	59	583
20	445	60	588
21	446	61	589
22	452	62	591
23	452	63	591
24	453	64	593
25	470	65	599
26	474	66	600
27	476	67	610
28	476	68	613
29	479	69	620
30	484	70	620
31	485	71	622
32	486	72	622
33	489	73	640
34	492	74	649
35	493	75	657
36	496	76	660
37	506	77	664
38	512	78	674
39	512	79	674
40	514	80	725



a Unbiased

Key

X Failure stress, σ, MPa

Y Probability of failure, P_f

Figure B.1 — Failure data in Clause B.1

B.2 Next, consider a sample that exhibits multiple active flaw distributions (see Table B.2). Here, each flexural test specimen was subjected to fractographic analysis. The failure origin was identified as either a volume or a surface fracture origin, and parameter estimates were obtained by using Equations 17 and 18. For the analysis with volume fracture origins, r = 13, and the calculations yielded values of $(\hat{m})_V = 6{,}79$ and $(\hat{\sigma}_\theta)_V$ = 876 MPa. For the analysis with surface fracture origins, r = 66, and the calculations yielded values of $(\hat{m})_A = 21{,}0$ and $(\hat{\sigma}_\theta)_A = 693$ MPa. For the most part, the data as plotted in Figure B.2 fall near the solid curve, which represents the combined probability of failure [7]:

$$P_f = 1 - \left[1 - \left(P_f\right)_A\right]\left[1 - \left(P_f\right)_V\right] \tag{B.1}$$

where $(P_f)_V$ is calculated by using

$$(P_f)_V = 1 - \exp\left\{-\left[\frac{\sigma}{(\hat{\sigma}_\theta)_V}\right]^{(\hat{m})_V}\right\} \tag{B.2}$$

and $(P_f)_A$ is calculated by using

$$(P_f)_A = 1 - \exp\left\{-\left[\frac{\sigma}{(\hat{\sigma}_\theta)_A}\right]^{(\hat{m})_A}\right\} \tag{B.3}$$

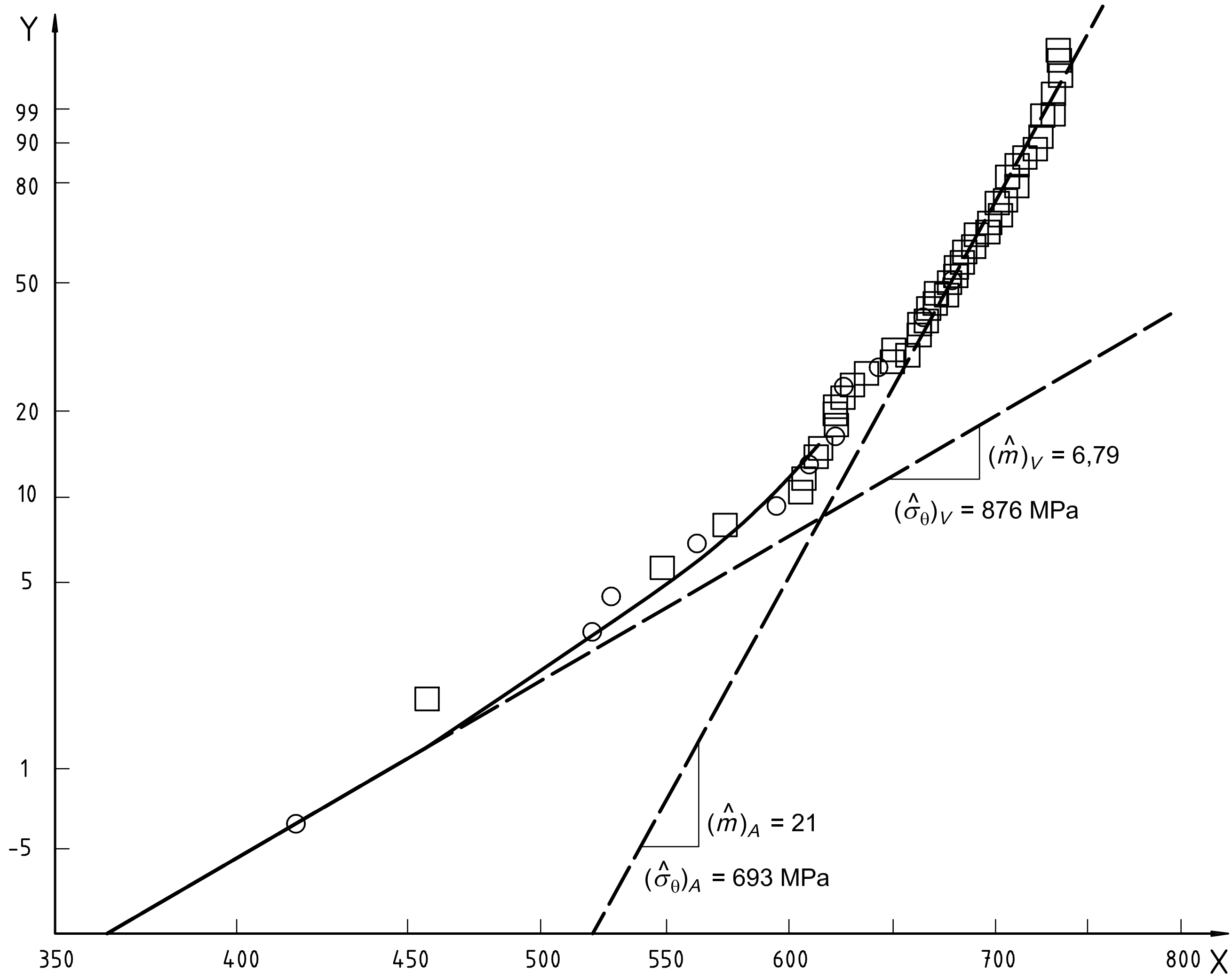
The curve obtained from Equation (B.1) asymptotically approaches the intersecting straight lines that are defined by the estimated parameters and calculated from Equations (B.2) and (B.3). Inserting the estimated Weibull parameters (obtained from the analysis for volume fracture origins) into Equation (A.3) along with the specimen geometry ($L_o = 40$ mm, $L_i = 20$ mm, $d = 3{,}5$ mm and $b = 4{,}5$ mm) yields $(\hat{\sigma}_0)_V = 65{,}6$ MPa$\cdot(m)^{0{,}442}$. Inserting the estimated Weibull parameters (obtained from the analysis for surface fracture origins) into Equation (A.5) yields $(\hat{\sigma}_0)_A = 446$ MPa$\cdot(m)^{0{,}95}$.

Table B.2 — Bimodal failure stress data — Example 2

Specimen number, N	**Strength, σ_f** MPa	**Fracture origin type** [a]	**Specimen number, N**	**Strength, σ_f** MPa	**Fracture origin type** [a]
1	416	V	41	671	S
2	458	S	42	672	S
3	520	V	43	672	S
4	527	V	44	674	S
5	546	S	45	677	S
6	561	V	46	677	S
7	572	S	47	678	S
8	595	V	48	680	S
9	604	S	49	683	S
10	604	S	50	684	S
11	609	V	51	686	S
12	612	S	52	687	S
13	614	S	53	687	S
14	621	V	54	691	S
15	622	S	55	694	S
16	622	S	56	695	S
17	622	V	57	700	S
18	622	S	58	703	S
19	625	S	59	703	S
20	626	V	60	703	S
21	631	S	61	703	S
22	640	S	62	704	S
23	643	V	63	704	S
24	649	S	64	706	S
25	650	S	65	710	S
26	652	V	66	713	S
27	655	S	67	716	S
28	657	S	68	716	S
29	657	V	69	716	S
30	660	S	70	716	S
31	660	S	71	716	S
32	662	V	72	717	S
33	662	S	73	725	S
34	662	S	74	725	S
35	664	S	75	725	S
36	664	S	76	726	S
37	664	S	77	727	S
38	666	S	78	729	S
39	669	S	79	732	S
40	671	S			

[a] Volume fracture origin, V; surface flaw origin, S.

B.3 It should be noted in this example that fractography apparently indicated that all volume failures were initiated from a single distribution of volume flaws, and that all surface failures were initiated from a single distribution of surface flaws. Often, fractography will indicate more complex situations such as two independent distributions of volume flaws (e.g., inclusions of foreign material and large voids) in addition to a distribution of surface flaws. Analysis of this type of sample would be very similar to the analysis discussed above, except that Equations 17 and 18 would be used three times instead of twice, and the resulting figure would include three straight lines labelled accordingly.



Key

X Failure stress, σ, MPa

Y Probability of failure, P_f

Figure B.2 — Failure data from Clause B.2

B.4 As an example of computing unbiased estimates of the Weibull modulus, and bounds on both the Weibull modulus and the Weibull characteristic strength, consider the unimodal failure sample presented in Clause B.1. The sample contained 80 specimens and the biased estimate of the Weibull modulus was determined to be $\hat{m} = 6{,}48$. The unbiasing factor corresponding to this sample size is UF = 0,984, which is obtained from Table 1 (see 5.5.2). Thus, the unbiased estimate of the Weibull modulus is given as

$$\hat{m}_U = \hat{m} \times \text{UF} \tag{B.4}$$

$= 6{,}48 \times 0{,}984$

$= 6{,}38$

The upper bound for this example is

$$\hat{m}_{\text{upper}} = \hat{m} / q_{0,05} \tag{B.5}$$

$= 6{,}48/0{,}878$

$= 7{,}38$

where $q_{0,05}$ is obtained from Table 2 for a sample size of 80 failed specimens. The lower bound is

$$\hat{m}_{\text{lower}} = \hat{m} / q_{0,95} \tag{B.6}$$

$= 6{,}48/1{,}173$

$= 5{,}52$

where $q_{0,95}$ is also obtained from Table 2.

Similarly, the upper bound on $\hat{\sigma}_\theta$ is

$$\left(\hat{\sigma}_\theta\right)_{\text{upper}} = \left(\hat{\sigma}_\theta\right) \exp\left(t_{0,05} / \hat{m}\right) \tag{B.7}$$

$= (556)\exp(0{,}197/6{,}48)$

$= 573$ MPa

where $t_{0,05}$ is obtained from Table 3 for a sample size of 80 failed specimens. The lower bound on $\hat{\sigma}_\theta$ is

$$\left(\hat{\sigma}_\theta\right)_{\text{lower}} = \left(\hat{\sigma}_\theta\right) \exp\left(-t_{0,95} / \hat{m}\right) \tag{B.8}$$

$= (556)\exp(-0{,}197/6{,}48)$

$= 539$ MPa

where $t_{0,95}$ is also obtained from Table 3.

Annex C
(informative)

Test specimens with unidentified fracture origin

C.1 The four options

C.1.1 General

Clause 6.2.2 described four options, a) to d), that the experimentalist can utilize when unidentified fracture origins are encountered during fractographic examination. C.1.2 to C.1.5 further define the four options, and use examples to illustrate appropriate and inappropriate situations for their use.

C.1.2 Option a)

Option a) involves using all available fractographic information to subjectively assign a specimen with an unidentified origin to a previously identified fracture origin classification. Many specimens with unidentified fracture origins have some fractographic information that was judged to be insufficient for positive identification and classification. (It should be noted that the degree of certainty required for "positive identification" of a fracture-initiating flaw varies from one fractographer to another.) In such cases, option a) permits the experimentalist the use of the incomplete fractographic information to assign the unidentified fracture origin to a previously identified flaw classification. This option is preferred when partial fractographic information is available. As an example, consider a tensile specimen where fractography was inconclusive. Fractographic markings may have indicated that the origin was located at or very near the specimen surface, but the fracture-initiating flaw could not be positively identified. Other specimens from the sample were positively identified as failing from machining flaws. It is recognized that machining damage is often difficult to discern therefore, in this case, it would be appropriate to use option a) and infer that the origin is machining damage. The test report should clearly indicate each specimen where this (or any other) option is used for classifying unidentified specimens. The conclusion of machining damage in this example, however, could be erroneous; e.g., the fracture initiating flaw may have been a "mainstream microstructural feature" [3], [12] (which is also typically difficult to resolve and identify) that happened to be located near the specimen surface. The possibility of erroneous classifications such as this are unavoidable in the absence of positive identification of fracture origins.

C.1.3 Option b)

Option b) involves assigning the unidentified fracture origin to that fracture origin classification of the test specimen closest in strength. The specimen closest in strength must have a positively identified fracture origin [not one assigned using options a) to d)]. As an example of use of this option, consider a tensile specimen that shattered upon failure such that the fracture origin was damaged and lost, but fracture was clearly initiated from an internal flaw. Other specimens from the sample included positive identification of inclusions and large pores as two active volume-distributed fracture origin classifications. When the fracture strengths from the total data set were ordered, the specimen closest in strength to the specimen with the unidentified fracture origin was the specimen that failed due to an inclusion. Use of option b) for this test specimen would then allow the unidentified origin to be classified as an inclusion. Justification for option b) arises from the tendency of concurrent (competing) flaw distributions to group together specimens with the same origin classification when the test specimens are listed in order of fracture strength. Therefore, the most likely fracture origin classification of a random unidentified specimen is the classification of the specimen closest in strength. The above example can be modified slightly to illustrate a situation where option b) would be inappropriate. If, the fracture origin classification of the specimen closest in strength was a machining flaw, then option b) would lead to a conclusion inconsistent with the fractographic observation that failure occurred from an internal flaw. Fractographic evidence should always supersede conclusions from option b).

C.1.4 Option c)

Option c) assumes that the unidentified fracture origins belong to a new, unclassified flaw type and treats these fracture origins as a separate flaw distribution in the censored data analysis. This may occur when the fractographer cannot recognize the flaw type because features of the flaw are particularly subtle and difficult to resolve. In such cases, the fractographer may consistently fail to locate and classify the fracture origin. Examples of flaw types that are difficult to identify include machining damage, zones of atypically high microporosity, and mainstream microstructural features. Option c) may be appropriate if a set of specimens with unidentified fracture origins have similar and apparently related features. Unfortunately there are many situations where option c) is incorrect and where use of this option could result in substantial errors in parameter estimates; e.g., consider the case where several unidentified specimens are concentrated in the upper tail (high strength) of the strength distribution. These fracture origins may belong to a classification that has been previously identified, but the smaller flaws at the origins were harder to locate, or possibly the origins were lost due to the greater fragmentation associated with high strength specimens. Use of option c) to treat these high strength specimens as a new flaw classification would create a bias error of unknown magnitude in the parameter estimates of the proper flaw classification.

C.1.5 Option d)

Option d) involves the removal of test specimens with unidentified fracture origins from the sample (i.e., the strengths are removed from the list of observed strengths). This option is rarely appropriate, and is not recommended by this International Standard unless there is clear justification. Option d) is only valid when test specimens with unidentified fracture origins are randomly distributed through the full range of strengths and flaw classifications. There are few plausible physical processes that create such a random selection. An example where option d) is justified is a data set of 50 specimens where the first 10 fractured specimens (in order of testing) were misplaced or destroyed after testing but prior to fractography. The unidentified specimens were therefore created by a process that is random. That is, the 10 strengths are expected to be randomly distributed through the strength distribution of the remaining 40, and the 10 origin classifications are expected to be randomly distributed through the origin types of the remaining 40. (In this example, option b) could also be considered.) Option d) is not appropriate where unidentified fracture origins are a consequence of high-strength test specimens shattering virulently such that the fragment with the origin is lost. This situation occurs with more frequency in the upper tail (high strength) of the strength distribution, and thus the unidentified fracture origins would not occur at random strengths.

C.2 Proper use and implementation of the four options

C.2.1 When partial fractographic information is available, option a) is preferred and should be used to incorporate the information as completely as possible into the assignment of fracture origin classification. Option d) should be used only in unusual situations where a random process for creation of unidentified origins can be justified.

C.2.2 Situations may arise where more than one option will be used within a single data set; e.g., of five specimens with unidentified origins, three might be classified based on partial fractographic information [option a)], while the remaining two, which have no fractographic hints, might then be classified using option b).

C.2.3 When specimens with unidentified fracture origins are contained within a data set, the test report (see Clause 8) should include a full description of which specimens were unidentified, and which option or options were used to classify the specimens.

C.2.4 If the unidentified fracture origins occur frequently in the lower tail of the strength distribution, then caution and extra attention is warranted. Strength analyses are typically extrapolated to lower strengths and lower probabilities of failure than those observed in the data set. Proper statistical evaluation and assignment of fracture origin classifications near the lower strength tail is therefore particularly important because the low strength distribution typically dominates extrapolations of this type.

C.2.5 When only a few fracture origins are unidentified, effects of incorrect classification are minimal. When more than 5 % or 10 % of the origins are unidentified, substantial statistical bias in estimates of parameters can result. When used for design applications, proper choice of options from C.1.2 to C.1.5 is critical and should be carefully justified in the test report. In such design applications, it may be prudent to carry out the analysis for more than one option to determine the sensitivity to choice of an improper option; e.g., in a group of 50 specimens with 10 unidentified origins (no partial fractographic information), the analysis could be conducted first using option b) then again using option c). The results from the two analyses could then be used individually to estimate the behaviour of the designed component. If a conservative prediction of component behaviour is warranted, the more conservative result of the two analyses should be used.

C.2.6 Finally, if most or all of the test specimens within a sample contain unidentified fracture origins, then censored data analysis in accordance with this International Standard is not possible. The strengths should be plotted on Weibull probability axes and, if the data reveal a pronounced bend (concave upwards) which is characteristic of two or more concurrent flaw distributions, then the methods described in this International Standard cannot be used without further refinements.

Annex D
(informative)

Fortran program

D.1 Using maximum likelihood estimators to compute estimates of the Weibull parameters requires solving Equations 17 and 18 for $\hat{m}$ and $\hat{\sigma}_\theta$, respectively. The solution of Equation 18 is straightforward once the estimate of the Weibull modulus $\hat{m}$ is obtained from Equation 17. Obtaining the root of Equation 17 requires an iterative numerical solution. In this annex the theoretical approach is presented for the numerical solution of these equations, along with the details of a computer algorithm (optional) that can be used to solve Equations 17 and 18.

D.2 The algorithm employs a Newton-Raphson technique to find the root of Equation 17. The root of Equation 17 represents a biased estimate of the Weibull modulus. Solution of Equation 18, which depends on the *biased* value of $\hat{m}$, is effectively an *unbiased* estimate of the characteristic strength. The reader is cautioned not to correct $\hat{m}$ for bias prior to computing the characteristic strength. This would yield an incorrect value of $\hat{\sigma}_\theta$. This approach expands Equation 17 in a Taylor series about $\hat{m}_0$:

$$f(\hat{m}) = f(\hat{m}_0) + (\hat{m} - \hat{m}_0)\left[f'(\hat{m}_0)\right] + \left[\frac{(\hat{m} - \hat{m}_0)^2}{2}\right] f''(\hat{m}_0) + \ldots \quad \text{(D.1)}$$

where $f(\hat{m})$ represents the right-hand side of Equation 17, and $\hat{m}_0$ is not a root of $f(\hat{m})$ but is reasonably close. Taking:

$$\Delta\hat{m} = \hat{m} - \hat{m}_0 \quad \text{(D.2)}$$

and setting Equation (D.1) equal to zero, then:

$$0 = f(\hat{m}_0) + (\Delta\hat{m})\left[f'(\hat{m}_0)\right] + \left[\frac{(\Delta\hat{m})^2}{2}\right] f''(\hat{m}_0) + \ldots \quad \text{(D.3)}$$

If the Taylor series expansion is truncated after the first three terms, the resulting expression is quadratic in $\Delta\hat{m}$. The roots of the quadratic form of Equation (D.3) are:

$$\Delta\hat{m}_{A,b} = -\left[\frac{f'(\hat{m})}{f''(\hat{m})}\right] \pm \left[\left(\frac{f'(\hat{m})}{f''(\hat{m})}\right)^2 - 2\left(\frac{f(\hat{m})}{f''(\hat{m})}\right)\right]^{1/2} \quad \text{(D.4)}$$

After obtaining $\Delta\hat{m}_{a,b}$ and knowing $\hat{m}_0$, Equation (D.2) is then solved for two values that represent improved (better than $\hat{m}_0$) estimates of the roots of $f(\hat{m})$, thus:

$$\hat{m}_A = \hat{m}_0 + \Delta\hat{m}_A \quad \text{(D.5)}$$

$$\hat{m}_b = \hat{m}_0 + \Delta\hat{m}_b \quad \text{(D.6)}$$

Equation 17 is evaluated with both values of $\hat{m}$ and the quantity that yields a smaller functional value is accepted as the updated estimate. This updated value of $\hat{m}$ replaces $\hat{m}_0$ in Equation (D.4), and the next iteration is performed. The iterative procedure is terminated when the functional evaluation of Equation 17 becomes less than some predetermined tolerance.

D.3 The following variable name list is provided as a convenience for interpreting the source code of the algorithm:

DF, DDF	first and second derivatives with respect to $\hat{m}$ of Equation 17
EPS	predetermined convergence criterion
F	function defined in Equation 17
NLIM	Maximum numbers of iterations allowed in determining the root
NSUSP	number of suspended (or censored) data (< NT)
NT	number of failure stresses
ST	failure stress; an argument passed to MAXL as input
STNORM	the largest failure stress; used to normalize all failure stresses to prevent computational overflows
MO	updated value of $\hat{m}$
MA, MB	values of the roots $\hat{m}_A$ and $\hat{m}_b$
WCS	estimated Weibull characteristic strength
WMT	maximum likelihood estimate of the Weibull modulus

D.4 The following is a listing of the FORTRAN source code for the algorithm discussed above.

```
C
C   *******************************************************************
C   *                                                                 *
C   *                                                                 *
C   *       THIS PROGRAM CALCULATES TWO PARAMETER MAXIMUM             *
C   *       LIKELIHOOD ESTIMATES FROM FAILURE DATA WITH AN            *
C   *       ASSUMED UNDERLYING WEIBULL DISTRIBUTION.  THE             *
C   *       ALGORITHM USES A NONLINEAR NEWTON-RAPHSON METHOD,         *
C   *       AND ACCOMODATES CENSORED DATA.                            *
C   *                                                                 *
C   *       REFERENCES:  "ADVANCED CALCULUS FOR APPLICATIONS"         *
C   *                     by HILDEBRAND                               *
C   *                     PRENTICE-HALL, INC.; 1962                   *
C   *                                                                 *
C   *                    "APPLIED LIFE DATA ANALYSIS"                 *
C   *                     by NELSON                                   *
C   *                     WILEY & SONS INC.; 1982                     *
C   *                                                                 *
C   *******************************************************************
C
      IMPLICIT REAL *8(A-H,O-Z)
      DOUBLE PRECISION ST(1000),ST1(1000)
      DOUBLE PRECISION M0, MA, MB, M1
      COMMON /DATA/ NFAIL, SUM1, NT, ST, ZERO, ONE
      ZERO = 0.D0
      ONE = 1.D0
      TWO = 2.D0
      EPS = 5.0D-10
      NLIM = 500
      M0 = 10.0
C
C   --- READ THE FAILURE DATA USING FREE FORMATS;
C        FILE CONTAINING FAILURE DATA IS ALLOCATED TO UNIT 8
C
      DO 10 I = 1,1000
        ST(I) = ZERO
        ST1(I) = ZERO
   10 CONTINUE
      STNORM = ZERO
      READ(8,*) NT
      READ(8,*) NSUSP
      NFAIL = NT - NSUSP
```

```
      DO 20 I = 1,NT
        READ(8,*) ST(I)
        STNORM = DMAX1(STNORM,ST(I))
 20   CONTINUE
C
C   --- NORMALIZE FAILURE DATA WITH LARGEST VALUE
C
      DO 30 I = 1,NT
        ST(I) = ST(I)/STNORM
 30   CONTINUE
C
      SUM1 = ZERO
      DO 40 I = 1,NFAIL
        READ(8,*) ST1(I)
        ST1(I) = ST1(I)/STNORM
        SUM1 = SUM1 + DLOG(ST1(I))
 40   CONTINUE
C
C   --- THE FUNCTION F IS DEFINED BY EQ 14 OF ASTM STANDARD C 1239
C
C   --- EVALUATE F(M0) AND THE ASSOCIATED SUMS WHICH ARE USED TO CALCULATE
C        THE DERIVATIVES OF F WITH RESPECT TO M
C
      CALL SUM (M0, SUM2, SUM3, F)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C   **********************************************************************
C   *                   NEWTON-RAPHSON ROOT SOLVER                         *
C   **********************************************************************
C
C   --- USE TAYLOR SERIES EXPANSION (INCLUDING SECOND DERIVATIVES)
C       FOUND ON PAGE 362 OF "ADVANCED CALCULUS FOR APPLICATIONS BY
C       HILDEBRAND (FIRST EDITION, FIFTH PRINTING) TO DETERMINE THE ROOTS
C       OF THE FOLLOWING EQUATION, WHICH IS QUADRATIC IN DELTA M.
C
C          F(M0+DELTA M) = 0
C                        = F(M0) + DELTA M * F'(M0)
C                            + (DELTA M)**2 * F''(M0)/2
C
C       HERE M0 IS THE CURRENT ESTIMATE OF M.
C       THE FORMULA YIELDS TWO ROOTS, DELTA MA AND DELTA MB.
C       MA AND MB ARE THE UPDATED VALUES OF M, WHERE
C
C                  M(A,B) = M0 + DELTA M(A,B)
C
C       F(MA) AND F(MB) ARE BOTH EVALUATED.  THE ESTIMATE THAT PRODUCES THE
C       SMALLEST ABSOLUTE VALUE OF F IS CHOSEN FOR THE NEXT ITERATION.
C
C       IF THE QUADRATIC EQUATION DOES NOT HAVE REAL ROOTS, AN
C       APPROXIMATE SOLUTION FOUND ON PAGE 363 OF HILDEBRAND IS USED, I.E.,
C
C            DELTA M = - (F(M0)/F'(M0)) *
C                        (1 + (DELTA M **2) * (F''(M0)/2*F(M0)))
C
C        WHERE ON THE RIGHT-HAND-SIDE OF THE EQN, DELTA M IS TAKEN AS THE
C        FIRST ORDER APPROXIMATION, DELTA M = -F(M0)/F'(M0)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      DO 60 K = 1,NLIM
C
C  --- CALCULATE THE FIRST AND SECOND DERIVATIVES OF THE FUNCTION F
C
        DSUM3 = ZERO
        DDSUM3 = ZERO
        DO 50 I = 1,NT
          DSUM3 = DSUM3+DLOG(ST(I))*(ST(I))**M0*DLOG(ST(I))
          DDSUM3 = DDSUM3 + (DLOG(ST(I)))**3*(ST(I))**M0
 50     CONTINUE
        DSUM2 = SUM3
        DDSUM2 = DSUM3
        DF = (SUM2 * DSUM3 - SUM3 * DSUM2)/(SUM2**2) + ONE/(M0**2)
C
        DDF = ((SUM2 * DDSUM3 - SUM3 * DDSUM2)/SUM2**2)
     $ - (TWO * DSUM2 * (SUM2 * DSUM3 - SUM3 * DSUM2)/SUM2**3)
     $ - TWO/M0**3

        RADICAL = (DF/DDF)**2 - TWO*F/DDF
        IF (RADICAL .GE. ZERO) THEN
C
C  --- CALCULATE THE ROOTS OF THE QUADRATIC EQUATION
C
          RADICAL = DSQRT(RADICAL)
          MA = M0 - (DF/DDF) + RADICAL
          MB = M0 - (DF/DDF) - RADICAL
C
C   --- CALCULATE F(MA), F(MB), AND THE ASSOCIATED SUMS
```

```
C
          CALL SUM (MA, SUM2A, SUM3A, FA)
          CALL SUM (MB, SUM2B, SUM3B, FB)
C
C   --- SELECT THE BETTER ROOT BY COMPARING THE ABSOLUTE
C          VALUE OF THE FUNCTION F
C
          IF (DABS(FA) .LE. DABS(FB)) THEN
            M0 = MA
            F = FA
            SUM2 = SUM2A
            SUM3 = SUM3A
          ELSE
            M0 = MB
            F = FB
            SUM2 = SUM2B
            SUM3 = SUM3B
          END IF

        ELSE
C
C   --- IF THE ROOTS ARE COMPLEX, USE THE APPROXIMATE SOLUTION
C
          M1 = M0 - (F/DF)*(ONE+F*DDF/(TWO*DF**2))
C
C   --- CALCULATE F(M1) AND ITS ASSOCIATED SUMS
C
          CALL SUM (M1, SUM2, SUM3, F)
          M0 = M1
        END IF
C
C   --- CONVERGENCE CRITERION:
C          COMPARE THE ABSOLUTE VALUE OF THE FUNCTION F
C          WITH A PRESELECTED TOLERANCE
C
        IF (DABS(F) .LE. EPS) GO TO 70
   60 CONTINUE
C
C   --- MAXIMUM NO. OF ITERATIONS REACHED BEFORE SATISFACTORY VALUE OF M FOUND
C
      WRITE(6,100) NLIM
      GO TO 999
C
C   --- SATISFACTORY ESTIMATE OF WEIBULL MODULUS ATTAINED
C
   70 WMT = M0
C
C   --- COMPUTE THE ESTIMATE OF THE WEIBULL CHARACTERISTIC STRENGTH (WCS)
C
      RWMT = 1.0/WMT
      WCS = ((SUM2/NFAIL)**RWMT)*STNORM
      WRITE(6,110) WMT
      WRITE(6,120) WCS
  100 FORMAT(/,2X,'NO SOLUTION FOUND AFTER ',I4,' ITERATIONS OF THE
NEWTON-RAPHSON METHOD',/)
  110 FORMAT(/,2X,' THE ESTIMATED WEIBULL MODULUS = ',F8.3,/)
  120 FORMAT(/,2X,' THE ESTIMATED CHARACTERISTIC STRENGTH = ',F8.3,/)
  999 CONTINUE
      STOP
      END

      SUBROUTINE SUM (M, SUM2, SUM3, F)
      IMPLICIT REAL*8 (A-H, O-Z)
      DOUBLE PRECISION ST(1000), M
      COMMON /DATA/ NFAIL, SUM1, NT, ST, ZERO, ONE
      SUM2 = ZERO
      SUM3 = ZERO
      DO 10 I = 1,NT
        SUM2 = SUM2 + ((ST(I))**M)
        SUM3 = SUM3 + (DLOG(ST(I)) * ((ST(I))**M))
  10  CONTINUE
      F = (SUM3/SUM2) - (SUM1/NFAIL) - (ONE/M)
      RETURN
      END
```

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