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Statistical interpretation of data —

Part 6:
Determination of statistical tolerance intervals

Interprétation statistique des données —

Partie 6: Détermination des intervalles statistiques de dispersion

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Foreword

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The committee responsible for this document is ISO/TC 69, *Applications of statistical methods*.

This second edition cancels and replaces the first edition (ISO 16269:2005), which has been technically revised.

ISO 16269 consists of the following parts, under the general title *Statistical interpretation of data*:

— *Part 4: Detection and treatment of outliers*

— *Part 6: Determination of statistical tolerance intervals*

— *Part 7: Median — Estimation and confidence intervals*

— *Part 8: Determination of prediction intervals*

Introduction

A statistical tolerance interval is an estimated interval, based on a sample, which can be asserted with confidence level $1 - \alpha$, for example 0,95, to contain at least a specified proportion p of the items in the population. The limits of a statistical tolerance interval are called statistical tolerance limits. The confidence level $1 - \alpha$ is the probability that a statistical tolerance interval constructed in the prescribed manner will contain at least a proportion p of the population. Conversely, the probability that this interval will contain less than the proportion p of the population is α. This part of ISO 16269 describes both one-sided and two-sided statistical tolerance intervals; a one-sided interval is constructed with an upper or a lower limit while a two-sided interval is constructed with both an upper and a lower limit.

A statistical tolerance interval depends on a confidence level $1 - \alpha$ and a stated proportion p of the population. The confidence level of a statistical tolerance interval is well understood from a confidence interval for a parameter. The confidence statement of a confidence interval is that the confidence interval contains the true value of the parameter a proportion $1 - \alpha$ of the cases in a long series of repeated random samples under identical conditions. Similarly the confidence statement of a statistical tolerance interval states that at least a proportion p of the population is contained in the interval in a proportion $1 - \alpha$ of the cases of a long series of repeated random samples under identical conditions. So if we think of the stated proportion of p of the population as a parameter, the idea behind statistical tolerance intervals is similar to the idea behind confidence intervals.

Statistical tolerance intervals are functions of the observations of the sample, i.e. statistics, and they will generally take different values for different samples. It is necessary that the observations be independent for the procedures provided in this part of ISO 16269 to be valid.

Two types of statistical tolerance interval are provided in this part of ISO 16269, parametric and distribution-free. The parametric approach is based on the assumption that the characteristic being studied in the population has a normal distribution; hence the confidence that the calculated statistical tolerance interval contains at least a proportion p of the population can only be taken to be $1 - \alpha$ if the normality assumption is true. For normally distributed characteristics, the statistical tolerance interval is determined using one of the Forms A, B, or C given in Annex B.

Parametric methods for distributions other than the normal are not considered in this part of ISO 16269. If departure from normality is suspected in the population, distribution-free statistical tolerance intervals may be constructed. The procedure for the determination of a statistical tolerance interval for any continuous distribution is provided in Form D of Annex B.

The statistical tolerance limits discussed in this part of ISO 16269 can be used to compare the natural capability of a process with one or two given specification limits, either an upper one U or a lower one L or both in statistical process management.

Above the upper specification limit U there is the upper fraction nonconforming p_U (ISO 3534-2:2006, 2.5.4) and below the lower specification limit L there is the lower fraction nonconforming p_L (ISO 3534-2:2006, 2.5.5). The sum $p_U + p_L = p_t$ is called the total fraction nonconforming. (ISO 3534-2:2006, 2.5.6). Between the specification limits U and L there is the fraction conforming $1 - p_t$.

The ideas behind statistical tolerance intervals are more widespread than is usually appreciated, for example in acceptance sampling by variables and in statistical process management, as will be pointed out in the next two paragraphs.

In acceptance sampling by variables, the limits U and/or L will be known, p_U , p_L or p_t will be specified as an acceptable quality limit (AQL), α will be implied and the lot is accepted if there is at least an implicit $100(1–\alpha)\%$ confidence that the AQL is not exceeded.

In statistical process management the limits U and L are fixed in advance and the fractions p_U, p_L and p_t are either calculated, if the distribution is assumed to be known, or otherwise estimated. This is an example of a quality control application, but there are many other applications of statistical tolerance intervals given in textbooks such as Hahn and Meeker.[13]

In contrast, for the statistical tolerance intervals considered in this part of ISO 16269, the confidence level for the interval estimator and the proportion of the distribution within the interval (corresponding to the fraction conforming mentioned above) are fixed in advance, and the limits are estimated. These limits may be compared with U and L. Hence the appropriateness of the given specification limits U and L can be compared with the actual properties of the process. The one-sided statistical tolerance intervals are used when only either the upper specification limit U or the lower specification limit L is relevant, while the two-sided intervals are used when both the upper and the lower specification limits are considered simultaneously.

The terminology with regard to these different limits and intervals has been confusing, as the "specification limits" were earlier also called "tolerance limits" (see the terminology standard ISO 3534-2:1993, 1.4.3, where both these terms as well as the term "limiting values" were all used as synonyms for this concept). In the latest revision of ISO 3534-2:2006, 3.1.3, only the term specification limits have been kept for this concept. Furthermore, the *Guide for the expression of uncertainty in measurement* [5] uses the term "coverage factor" defined as a "numerical factor used as a multiplier of the combined standard uncertainty in order to obtain an expanded uncertainty". This use of "coverage" differs from the use of the term in this part of ISO 16269.

The first edition of this standard gave extensive tables of the factor k for one-sided and two-sided tolerance intervals when the mean is unknown but the standard deviation is known. In this second edition of the standard those tables are omitted. Instead, exact k-factors are given in Annex A when one of the parameters of the normal distribution is unknown and the other parameter is known.

The first edition of this standard considered statistical tolerance intervals based only on a single sample of size n. This edition considers statistical tolerance intervals for m populations with the same standard deviation, based on samples from each of the m populations, each sample being of the same size n.

Statistical interpretation of data —

Part 6: Determination of statistical tolerance intervals

1 Scope

This part of ISO 16269 describes procedures for establishing statistical tolerance intervals that include at least a specified proportion of the population with a specified confidence level. Both one-sided and two-sided statistical tolerance intervals are provided, a one-sided interval having either an upper or a lower limit while a two-sided interval has both upper and lower limits. Two methods are provided, a parametric method for the case where the characteristic being studied has a normal distribution and a distribution-free method for the case where nothing is known about the distribution except that it is continuous. There is also a procedure for the establishment of two-sided statistical tolerance intervals for more than one normal sample with common unknown variance.

2 Normative references

The following documents, in whole or in part, are normatively referenced in this document and are indispensable for its application. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

ISO 3534-1:2006, *Statistics — Vocabulary and symbols — Part 1: General statistical terms and terms used in probability*

ISO 3534-2:2006, *Statistics — Vocabulary and symbols — Part 2: Applied statistics*

3 Terms, definitions and symbols

For the purposes of this document, the terms and definition given in ISO 3534-1, ISO 3534-2 and the following apply.

3.1 Terms and definitions

3.1.1
statistical tolerance interval
interval determined from a random sample in such a way that one may have a specified level of confidence that the interval covers at least a specified proportion of the sampled population

[SOURCE: ISO 3534-1:2006, 1.26]

Note 1 to entry: The confidence level in this context is the long-run proportion of intervals constructed in this manner that will include at least the specified proportion of the sampled population.

3.1.2
statistical tolerance limit
statistic representing an end point of a statistical tolerance interval

[SOURCE: ISO 3534-1:2006, 1.27]

Note 1 to entry: Statistical tolerance intervals may be either

— one-sided (with one of its limits fixed at the natural boundary of the random variable), in which case they have either an upper or a lower statistical tolerance limit, or

— two-sided, in which case they have both.

3.1.3
coverage
proportion of items in a population lying within a statistical tolerance interval

Note 1 to entry: This concept is not to be confused with the concept *coverage factor* used in the *Guide for the expression of uncertainty in measurement* (GUM)[5].

3.1.4
normal population
normally distributed population

3.2 Symbols

For the purposes of this part of ISO 16269, the following symbols apply.

$k_1(n; p; 1-\alpha)$	factor used to determine the limits of one-sided intervals i.e. x_L or x_U when μ is known and σ is unknown
$k_2(n; p; 1-\alpha)$	factor used to determine the limits of two-sided intervals i.e. x_L and x_U when μ is known and σ is unknown
$k_3(n; p; 1-\alpha)$	factor used to determine the limits of one-sided intervals i.e. x_L or x_U when μ is unknown and σ is known
$k_4(n; p; 1-\alpha)$	factor used to determine the limits of two-sided intervals i.e. x_L and x_U when μ is unknown and σ is unknown
$k_C(n; p; 1-\alpha)$	factor used to determine x_L or x_U when the values of μ and σ are unknown for one-sided statistical tolerance interval. The suffix C is chosen because this k-factor is tabulated in Annex C.
$k_D(n; m; p; 1-\alpha)$	factor used to determine x_{Li} and x_{Ui} (i = 1,2,...,m; $m \geq 2$) when the values of the means μ_i and the value of the common σ are unknown for the m two-sided statistical tolerance intervals. The suffix D is chosen because this k-factor is tabulated in Annex D.
n	number of observations in the sample
p	minimum proportion of the population asserted to be lying in the statistical tolerance interval
u_p	p-fractile of the standardized normal distribution
x_j	jth observed value ..
x_{ij}	jth observed value (j = 1,2,...,n) of ith sample (i = 1,2,...,m)
x_{max}	maximum value of the observed values: $x_{max} = \max\{x_1, x_2, ..., x_n\}$
x_{min}	minimum value of the observed values: $x_{min} = \min\{x_1, x_2, ..., x_n\}$
x_L	lower limit of the statistical tolerance interval
x_U	upper limit of the statistical tolerance interval
$\bar{x}$	sample mean, $\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$

$\bar{x}_i$ sample mean of ith sample, $(i=1,2,...,m)$, $\bar{x}_i=\frac{1}{n}\sum_{j=1}^{n}x_{ij}$

s sample standard deviation, $s=\sqrt{\frac{1}{n-1}\sum_{j=1}^{n}\left(x_j-\bar{x}\right)^2}=\sqrt{\frac{n\sum_{j=1}^{n}x_j^2-\left(\sum_{j=1}^{n}x_j\right)^2}{n(n-1)}}$

s_i sample standard deviation of ith sample, $(i=1,2,...,m)$, $s_i=\sqrt{\frac{1}{(n-1)}\sum_{j=1}^{n}(x_{ij}-\bar{x}_i)^2}$

s_P pooled sample standard deviation, $s_P=\sqrt{\frac{1}{m(n-1)}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}-\bar{x}_i)^2}=\sqrt{\frac{1}{m}\sum_{i=1}^{m}s_i^2}$

$1-\alpha$ confidence level for the assertion that the proportion of the population lying within the tolerance interval is greater than or equal to the specified level p

μ population mean

μ_i population mean of the ith population (i = 1,2,...,m)

σ population standard deviation

4 Procedures

4.1 Normal population with known mean and known variance

When the values of the mean, μ, and the variance, σ^2, of a normally distributed population are known, the distribution of the characteristic under investigation is fully determined. There is exactly a proportion p of the population:

a) to the right of $x_L = \mu - \mu_p \times \sigma$ (one-sided interval);

b) to the left of $x_U = \mu + \mu_p \times \sigma$ (one-sided interval);

c) between $x_L = \mu - \mu_{(1+p)/2} \times \sigma$ and $x_U = \mu + \mu_{(1+p)/2} \times \sigma$ (two-sided interval).

In the above equations, μ_p is the p-fractile of the standardized normal distribution.

NOTE As such statements are known to be true, they are made with 100 % confidence.

4.2 Normal population with unknown mean and known variance

When one or both parameters of the normal distribution are unknown but estimated from a random sample, intervals with similar properties to the ones in 4.1 can still be constructed. Suppose for example that the mean is unknown but the variance is known. Then a constant k can be found such that the interval between

$$x_L=\bar{x}-k\sigma \text{ and } x_U=\bar{x}+k\sigma$$

contains *at least* a proportion p of the population with a specified confidence of $1-\alpha$. Note two important distinctions from the situation in 4.1 where the parameters were assumed known. First, when one or more parameters are estimated the interval contains *at least* a proportion p of the population, not exactly a proportion p of the population. Secondly, when parameters are estimated, the statement is only true with a pre-specified confidence of $1-\alpha$. The factor k in the expression of the limits above depends on the unknown parameters of the normal distribution, on the proportion p, on the confidence coefficient $1-\alpha$, and on the number of observations in the random sample. Exact k-factors are given in Annex A when one of the parameters of the normal distribution is unknown and the other parameter is known.

4.3 Normal population with unknown mean and unknown variance

Forms A and B, given in Annex B, are applicable to the case where both the mean and the variance of the normal population are unknown. Form A applies to the one-sided case, while Form B applies to the two-sided case. Form A is used with the tables of k-factors in Annex C, or alternatively using the exact formula for the k-factor given in clause A.5 in Annex A. Form B is used with the k-factors given in the first column of the tables of Annex D. Details about the derivation of the k-factors of Annex D are given in Annex F.

4.4 Normal populations with unknown means and unknown common variance

Form C, given in Annex B, is applicable to the case where both the means and the variances of the normal populations are unknown. Furthermore, the variances are assumed to be identical for all populations under consideration, in which case we talk of the common variance.

4.5 Any continuous distribution of unknown type

If the characteristic under investigation is a variable from a population of unknown form, then a statistical tolerance interval can be determined from the sample order statistics $x_{(i)}$ of a sample of n independent random observations. The procedure given in Form D used in conjunction with Tables E.1 and E.2 provides the steps for the determination of the required sample size based on the order statistics to be used, the desired confidence level, and the desired content.

NOTE 1 Statistical tolerance intervals where the choice of end points (based on order statistics) does not depend on the sampled population are called *distribution-free* statistical tolerance intervals.

NOTE 2 This International Standard does not provide procedures for distributions of known type other than the normal distribution. However, if the distribution is continuous, the distribution-free method may be used. Selected references to scientific literature that may assist in determining tolerance intervals for other distributions are also provided at the end of this document.

5 Examples

5.1 Data for Examples 1 and 2

Forms A to B, given in Annex B, are illustrated by Examples 1 and 2 using the numerical values of ISO 2854:1976 [2], Clause 2, paragraph 1 of the introductory remarks, Table X, yarn 2: 12 measures of the breaking load of cotton yarn. It should be noted that the number of observations, $n = 12$, given here for these examples is considerably lower than the one recommended in ISO 2602 [1]. The numerical data and calculations in the different examples are expressed in centinewtons (see Table 1).

Table 1 — Data for Examples 1 and 2

Values in centinewtons

x	228,6	232,7	238,8	317,2	315,8	275,1	222,2	236,7	224,7	251,2	210,4	270,7

These measurements were obtained from a batch of 12000 bobbins, from one production job, packed in 120 boxes each containing 100 bobbins. Twelve boxes have been drawn at random from the batch and a bobbin has been drawn at random from each of these boxes. Test pieces of 50 cm length have been cut from the yarn on these bobbins, at about 5 m distance from the free end. The tests themselves have been carried out on the central parts of these test pieces. Previous information makes it reasonable to assume that the breaking loads measured in these conditions have virtually a normal distribution. It is demonstrated in ISO 2854; 1976 that the data do not contradict the assumption of a normal distribution.

By using the box plot graphical test of outliers given in ISO 16269-4, one can also conclude that none of the data values can be declared as outlier with significance level $\alpha = 0{,}05$.

The data in Table 1 give the following results:

Sample size: $n = 12$

Sample mean: $\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j = 3\,024{,}1/12 = 252{,}01$

Sample standard deviation: $s = \sqrt{\frac{n\sum_{j=1}^{n} x_j^2 - \left(\sum_{j=1}^{n} x_j\right)^2}{n(n-1)}} = \sqrt{\frac{166\,772{,}27}{12\times 11}} = \sqrt{1\,263{,}426\,3} = 35{,}545$

The formal presentation of the calculations will be given in Example 1 using Form A in Annex B (one-sided interval, unknown variance and unknown mean).

5.2 Example 1: One-sided statistical tolerance interval with unknown variance and unknown mean

A limit x_L is required such that it is possible to assert with confidence level $1 - \alpha = 0{,}95$ (95 %) that at least 0,95 (95 %) of the breaking loads of the items in the batch, when measured under the same conditions, are above x_L. The presentation of the results is given in detail below.

Determination of the statistical tolerance interval of proportion p:
a) one-sided interval “to the right”
Determined values:
b) proportion of the population selected for the statistical tolerance interval: $p = 0{,}95$
c) chosen confidence level: $1 - \alpha = 0{,}95$
d) sample size: $n = 12$
Value of tolerance factor from Table C.2 : $k_C(n; p; 1-\alpha) = 2{,}736\,4$

Calculations:
$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j =$ 252,01 $s = \sqrt{\frac{n\sum_{j=1}^{n} x_j^2 - \left(\sum_{j=1}^{n} x_j\right)^2}{n(n-1)}}$ = 35,545 $k_C(n; p; 1-\alpha) \times s = 97{,}265\,3$
Results: one-sided interval "to the right" The tolerance interval which will contain at least a proportion p of the population with confidence level $1 - \alpha$ has a lower limit: $x_L = \bar{x} - k_C(n; p; 1-\alpha) \times s = 154{,}7$

5.3 Example 2: Two-sided statistical tolerance interval under unknown mean and unknown variance

Suppose it is required to calculate the limits x_L and x_U such that it is possible to assert with a confidence level $1 - \alpha = 0{,}95$ that in a proportion of the batch at least equal to $p = 0{,}90$ (90 %) the breaking load falls between x_L and x_U.

The column with $m = 1$ and the row with $n = 12$ in Table D.4 gives

$$k_D(n;1;p;1-\alpha) = 2{,}6703$$

whence

$$x_L = \bar{x} - k_D(n;1;p;1-\alpha) \times s = 252{,}01 - 2{,}6703 \times 35{,}545 = 157{,}0$$

$$x_U = \bar{x} + k_D(n;1;p;1-\alpha) \times s = 252{,}01 + 2{,}6703 \times 35{,}545 = 347{,}0$$

5.4 Data for Examples 3 and 4

Suppose the percentage of solids in each of four batches of wet brewer's yeast, each from a different supplier, is to be determined. The percentages of the four batches are normally distributed with unknown means μ_i $i = 1,2,3,4$. From previous experience of these suppliers, it may be assumed that the variances are the same. A test for the following data gives no reason to suppose otherwise. The data are therefore assumed to have a common variance σ^2. The researcher wants to determine two-sided statistical tolerance intervals for the percentages of solids in each batch.

The values of random samples of size $n = 10$ from four batches [14] are given in Table 2:

Table 2 — Data for Examples 3 and 4

Values in percent

i	j									
	1	2	3	4	5	6	7	8	9	10
1	20	18	16	21	19	17	20	16	19	18
2	19	14	17	13	10	16	14	12	15	11
3	11	12	14	10	8	10	13	9	12	8
4	10	7	11	9	6	11	8	12	13	14

Notice that the jth value of the ith sample is denoted: x_{ij}.

These results yield the following:

Sample size: $n = 10$

Number of samples: $m = 4$

Sample means of each of the four batches:

$\bar{x}_1 = 184/10 = 18{,}4$; $\bar{x}_2 = 141/10 = 14{,}1$; $\bar{x}_3 = 107/10 = 10{,}7$; $\bar{x}_4 = 101/10 = 10{,}1$

Sample variances of each of the four batches:

$$s_1^2 = \frac{n\sum_{j=1}^{n} x_{1j}^2 - \left(\sum_{j=1}^{n} x_{1j}\right)^2}{n(n-1)} = \frac{264}{10\times 9} = 2{,}9333 \ ; \quad s_2^2 = \frac{n\sum_{j=1}^{n} x_{2j}^2 - \left(\sum_{j=1}^{n} x_{2j}\right)^2}{n(n-1)} = \frac{689}{10\times 9} = 7{,}6556$$

$$s_3^2 = \frac{n\sum_{j=1}^{n} x_{3j}^2 - \left(\sum_{j=1}^{n} x_{3j}\right)^2}{n(n-1)} = \frac{381}{10\times 9} = 4{,}2333 \ ; \quad s_4^2 = \frac{n\sum_{j=1}^{n} x_{4j}^2 - \left(\sum_{j=1}^{n} x_{4j}\right)^2}{n(n-1)} = \frac{609}{10\times 9} = 6{,}7667$$

Pooled sample standard deviation:

$$s_\text{P} = \sqrt{\frac{1}{m}\sum_{i=1}^{m} s_i^2} = \sqrt{\frac{1}{4}(2{,}9333 + 7{,}6556 + 4{,}2333 + 6{,}7667)} = 2{,}3232$$

Degrees of freedom of the pooled standard deviation:

$f = m(n - 1) = nm - m = 36$

5.5 Example 3: One-sided statistical tolerance intervals for separate populations with unknown common variance

Suppose it is desired to calculate lower statistical tolerance intervals for the four suppliers, i.e. it is desired to calculate intervals that contain at least a proportion p for all suppliers. Table C cannot provide the answer but the intervals are of the same form as was given in Example 1, namely a constant multiplied by the estimated standard deviation and subtracted from the estimated mean

$$x_{\mathrm{L}i} = \overline{x}_i - k(n_i; f; p; 1-\alpha) \times s_{\mathrm{P}},$$

where the constant $k(n_i;f;p;1-\alpha)$ depends on the size of the ith sample and the degrees of freedom of the pooled standard deviation. The expression for the constant is derived in Clause A.5 in Annex A, see Formula (A.14);

$$k(n_i; f; p; 1-\alpha) = \frac{1}{\sqrt{n_i}} t_{1-\alpha}(\sqrt{n_i} u_p; f),$$

where $t_{1-\alpha}(\sqrt{n_i} u_p; f)$ denotes the $1-\alpha$ quantile of the non-central t-distribution with non-centrality parameter $\sqrt{n_i} u_p$ and f degrees of freedom. The non-central t-distribution and in particular its quantiles are available in statistical software packages. Suppose a proportion $p = 0{,}95$ and a confidence coefficient $1 - \alpha = 0{,}95$ is desired. In this case $n_i = 10$ and $f = m(n - 1) = nm - m = 36$, so the constant is

$$k(10;36;0{,}95;0{,}95) = \frac{1}{\sqrt{10}} t_{0{,}95}(\sqrt{10} \times 1{,}6449;36) = 2{,}3471,$$

where the 0,95 quantile of the standardized normal distribution $u_{0{,}95} = 1{,}6449$ is inserted.

The values provided in the tables in Annex C are the special cases where the degrees of freedom are equal to the sample size minus 1 which is the degrees of freedom of the standard deviation based on a single sample of size n

$$k_{\mathrm{C}}(n;p;1-\alpha) = k(n;n-1;p;1-\alpha) = \frac{1}{\sqrt{n}} t_{1-\alpha}(\sqrt{n} u_p; n-1),$$

i.e. the special case, where the degrees of freedom of the estimate of the variance is $n - 1$.

It follows that the one-sided statistical tolerance limits computed for all four batches are as follows.

First batch: $x_{\mathrm{L}1} = \overline{x}_1 - k(n_1;\nu;p;1-\alpha) \times s_{\mathrm{P}} = 18{,}40 - 2{,}3471 \times 2{,}3232 = 12{,}94$

Second batch: $x_{\mathrm{L}2} = \overline{x}_2 - k(n_2;\nu;p;1-\alpha) \times s_{\mathrm{P}} = 14{,}10 - 2{,}3471 \times 2{,}3232 = 8{,}64$

Third batch: $x_{\mathrm{L}3} = \overline{x}_3 - k(n;\nu;p;1-\alpha) \times s_{\mathrm{P}} = 10{,}70 - 2{,}3471 \times 2{,}3232 = 4{,}66$

Fourth batch: $x_{\mathrm{L}4} = \overline{x}_4 - k(n;\nu;p;1-\alpha) \times s_{\mathrm{P}} = 10{,}10 - 2{,}3471 \times 2{,}3232 = 4{,}06$

If the upper statistical tolerance limits had been required, the same quantities would be combined except that the constant times the standard error would be added to the estimated mean.

5.6 Example 4: Two-sided statistical tolerance intervals for separate populations with unknown common variance

Case 1 — Computation for all batches ($m = 4$)

Table D.5 in Annex D gives for $n = 10$, $m = 4$, $f = m(n - 1) = 4(10 - 1) = 36$, $p = 0{,}95$ and $1 - \alpha = 0{,}95$ and the value of the two-sided statistical tolerance factor for unknown common variability σ^2 as

$$k_{\mathrm{D}}(n;m;p;1-\alpha) = 2{,}5964.$$

It follows that the two-sided statistical tolerance limits computed simultaneously for all batches are as follows.

First batch:

$$x_{L1} = \bar{x}_1 - k_D(n;m;p;1-\alpha) \times s_P = 18{,}40 - 2{,}5964 \times 2{,}3232 = 12{,}36$$

$$x_{U1} = \bar{x}_1 + k_D(n;m;p;1-\alpha) \times s_P = 18{,}40 + 2{,}5964 \times 2{,}3232 = 24{,}44$$

Second batch:

$$x_{L2} = \bar{x}_2 - k_D(n;m;p;1-\alpha) \times s_P = 14{,}10 - 2{,}5964 \times 2{,}3232 = 8{,}06$$

$$x_{U2} = \bar{x}_2 + k_D(n;m;p;1-\alpha) \times s_P = 14{,}10 + 2{,}5964 \times 2{,}3232 = 20{,}14$$

Third batch:

$$x_{L3} = \bar{x}_3 - k_D(n;m;p;1-\alpha) \times s_P = 10{,}70 - 2{,}5964 \times 2{,}3232 = 4{,}66$$

$$x_{U3} = \bar{x}_3 + k_D(n;m;p;1-\alpha) \times s_P = 10{,}70 + 2{,}5964 \times 2{,}3232 = 16{,}74$$

Fourth batch:

$$x_{L4} = \bar{x}4 - k_D(n;m;p;1-\alpha) \times s_P = 10{,}10 - 2{,}5964 \times 2{,}3232 = 4{,}06$$

$$x_{U4} = \bar{x}_4 + k_D(n;m;p;1-\alpha) \times s_P = 10{,}10 + 2{,}5964 \times 2{,}3232 = 16{,}14$$

NOTE The lower limits have been rounded down and the upper limits have been rounded up (in the second decimal place) to maintain the integrity of the confidence statements.

Case 2 — Individual computation for each batch (m = 1)

It is possible to compute these tolerance limits separately for each batch. For n = 10, m = 1, $f = m(n-1) = 1(10-1) = 9$, p = 0,95 and $1-\alpha$ = 0,95, the value of the two-sided statistical tolerance factor for unknown common variability σ^2 equals

$$k_D(10;1;0{,}95;0{,}95) = 3{,}3935$$

and can be found in Annex D (Table D.4).

Sample standard deviations of four batches:

$$s_1 = \sqrt{s_1^2} = \sqrt{2{,}9333} = 1{,}7127\ ; \qquad s_2 = \sqrt{s_2^2} = \sqrt{7{,}6556} = 2{,}7669$$

$$s_3 = \sqrt{s_3^2} = \sqrt{4{,}2333} = 2{,}0575\ ; \qquad s_4 = \sqrt{s_4^2} = \sqrt{6{,}7667} = 2{,}6013$$

Hence the two-sided statistical tolerance limits are as follows:

First batch:

$$x_{L1} = \bar{x}_1 - k_D(n;m;0{,}95;0{,}95) \times s_1 = \bar{x}_1 - k_D(10;1;0{,}95;0{,}95) \times s_1$$
$$= 18{,}40 - 3{,}3935 \times 1{,}7127 = 12{,}58$$
$$x_{U1} = \bar{x}_1 + k_D(n;m;p;1-\alpha) \times s_1 = \bar{x}_1 + k_D(10;1;0{,}95;0{,}95) \times s_1$$
$$= 18{,}40 + 3{,}3935 \times 1{,}7127 = 24{,}22$$

Second batch:

$$x_{L2} = \bar{x}_2 - k_D(n;m;p;1-\alpha)\times s_2 = \bar{x}_2 - k_D(10;1;p;1-\alpha)\times s_2$$
$$= 14,10 - 3,3935\times 2,7669 \quad = 4,70$$
$$x_{U2} = \bar{x}_2 + k_D(n;m;p;1-\alpha)\times s_2 = \bar{x}_2 + k_D(10;1;p;1-\alpha)\times s_2$$
$$= 14,10 + 3,3935\times 2,7669 \quad = 23,50$$

Third batch:

$$x_{L3} = \bar{x}_3 - k_D(n;m;p;1-\alpha)\times s_3 = \bar{x}_3 - k_D(n;m;p;1-\alpha)\times s_3$$
$$= 10,70 - 3,394\times 2,0575 \quad = 3,71$$
$$x_{U3} = \bar{x}_3 + k_D(n;m;p;1-\alpha)\times s_3 = \bar{x}_3 + k_D(10;1;p;1-\alpha)\times s_3$$
$$= 10,70 + 3,3935\times 2,0575 \quad = 17,69$$

Fourth batch:

$$x_{L4} = \bar{x}_4 - k_D(n;m;p;1-\alpha)\times s_4 = \bar{x}_4 - k_D(10;1;p;1-\alpha)\times s_4$$
$$= 10,10 - 3,3935\times 2,6013 \quad = 1,27$$
$$x_{U4} = \bar{x}_4 + k_D(n;m;p;1-\alpha)\times s_4 = \bar{x}_4 + k_D(10;1;p;1-\alpha)\times s_4$$
$$= 10,10 + 3,3935\times 2,6013 \quad = 18,93$$

When comparing the result of both cases it can be declared that the statistical tolerance intervals for batches 2, 3 and 4 are substantially smaller in Case 1 than in Case 2. But the statistical tolerance interval for the first batch is only a little larger in Case 2. The explanation is that the constant k_D is smaller in Case 1 than in Case 2 because the degrees of freedom are larger in Case 1. Batch 1 has the smallest estimated standard deviation and this compensates for the increase in the constant k_D.

We can conclude that the statistical tolerance intervals computed simultaneously for several populations can yield intervals shorter than the statistical tolerance intervals computed for each random sample separately, provided that the underlying normal populations have the same variance. This nice property follows from the fact that on the average, the estimate of the variance computed from several random samples is ’better’ than the estimate computed from one random sample, because the latter is based on a smaller number of observations.

5.7 Example 5: Any distribution of unknown type

Assume we have a sample, $x_1, x_2, \ldots, x_n$, of independent random observations on a population (continuous, discrete, or mixed) and let its order statistics be $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$.

It is possible to determine the sample size necessary to achieve at least 100(1 −α) % confidence that at least 100p % of the population are lying between the vth smallest observation (i.e., order statistic $x_{(v)}$) and the wth largest observation (i.e., order statistic $x_{(n-w+1)}$).

1) Determine the sample size n necessary to achieve at least 95 % confidence that at least 99 % of the population’s measured values lie between the minimum and maximum observations, i.e. between the first (v = 1) and the nth (w = 1) sample order statistics.

Based on the above description $v + w = 2$, $p = 0,99$, and $1 - \alpha = 0,95$. The minimum sample size determined from Table E.1 is 473 (the actual confidence level is 95,020 %). A few examples are given below.

2) Determine the sample size n necessary to achieve at least 95 % confidence that at least 95 % of the population’s measured values are greater than or equal to the minimum sample order statistic (v = 1 and w = 0).

Based on the above description, $v + w = 1$, $p = 0{,}95$, and $1 - \alpha = 0{,}95$. The minimum sample size determined from Table E.1 is 59 (the actual confidence level is 95,151 %).

3) Determine the sample size n necessary to achieve at least 95 % confidence that at least 99 % of the population's units are acceptable with at most one permissible nonconforming unit in the sample.

Based on the description in Annex G, $v + w = 2$ ($v + w - 1 = 1$ because 1 is the maximum permissible number of nonconforming items in the sample), $p = 0{,}99$, and $1 - \alpha = 0{,}95$. The minimum sample size determined from Table E.1 is 473 (the actual confidence level is 95,020 %). Note that this result is identical to that of the first example in this section.

4) Suppose that the distribution of X is expected to have long tails (i.e., produces occasional extreme positive and negative values) and extra measures are considered necessary to ensure the resulting statistical tolerance interval is of a useful length. The experimenter decides to exclude lower and upper order statistics such that the statistical tolerance interval is constructed between the fifth smallest ($v = 5$) and fifth largest ($w = 5$) order statistics. Determine the sample size n necessary to achieve at least 90 % confidence that at least 99 % of the population's measured values lie within this interval.

Based on the description in Annex G, $v + w = 10$, $p = 0{,}99$, and $1 - \alpha = 0{,}90$. The minimum sample size determined from Table E.1 is 1418 (the actual confidence level is 90,000 %) and the associated order statistics are $x_{(5)}$ and $x_{(1414)}$.

Annex A
(informative)

Exact k-factors for statistical tolerance intervals for the normal distribution

Annex A gives the exact k-factors for calculating tolerance intervals based on a single normal sample. In this annex, a sample of size n from the N(μ, σ) distribution is considered. Let $\bar{x}$ and s denote the sample mean and the sample standard deviation, respectively. Initially, we assume that $\bar{x}$ and s are estimated from the same sample, and in that case the x^2-distribution of $(n - 1)s^2/\sigma^2$ has $n - 1$ degrees of freedom. But we might have an independent estimate of the standard deviation with degrees of freedom f, where typically f is greater than $n - 1$. For example, this would be the case if the estimate of the standard deviation were based on several independent samples with a common standard deviation. The exact formulas are easily modified to deal with this situation.

Type of interval	Mean	Standard deviation	Symbol
One-sided	Known	Unknown	$k_1(n;p;1 - \alpha)$
Two-sided	Known	Unknown	$k_2(n;p;1 - \alpha)$
One-sided	Unknown	Known	$k_3(n;p;1 - \alpha)$
Two-sided	Unknown	Known	$k_4(n;p;1 - \alpha)$
One-sided	Unknown	Unknown	$k_C(n;p;1 - \alpha)$

A.1 One-sided statistical tolerance interval with known mean and unknown standard deviation

The interval $[-\infty, \mu + u_p\sigma]$ contains a proportion p of the population, and if

$$\mu + ks > \mu + u_p\sigma,$$

then the interval $[-\infty, \mu + ks]$ will contain a proportion of the population that is larger than p. We want to determine k such that this happens with the probability $1 - \alpha$, i.e.

$$P(\mu + ks > \mu + u_p\sigma) = P\left(\frac{s}{\sigma} > \frac{u_p}{k}\right) = 1 - \alpha \quad \text{(A.1)}$$

The distribution of s^2/σ^2 is $\chi^2/(n - 1)$ with $n - 1$ degrees of freedom, so it follows from the last equality in Formula (A.1) that

$$\frac{u_p}{k} = \sqrt{\frac{\chi_\alpha^2(n-1)}{n-1}}$$

so

$$k = u_p \frac{\sqrt{n-1}}{\sqrt{\chi_\alpha^2(n-1)}} \quad \text{(A.2)}$$

Here $\chi^2_\alpha(n-1)$ is the α fractile of the χ^2 distribution with $n - 1$ degrees of freedom, so this is a value that is exceeded with probability $1 - \alpha$ by the random variable $s^2(n - 1)/\sigma^2$.

The variable k in Formula (A.2) is $k_1(n;p;1 - \alpha)$.

A.2 Two-sided statistical tolerance interval with known mean and unknown standard deviation

The interval $[\mu + u_{\frac{1-p}{2}}\sigma, \mu + u_{\frac{1+p}{2}}\sigma]$ contains a proportion p of the population, and if

$$\mu + ks > \mu + u_{\frac{1+p}{2}}\sigma,$$

then the interval $[\mu - ks, \mu + ks]$ will contain a proportion of the population that is larger than p. We want to determine k such that this happens with the probability $1 - \alpha$, i.e.

$$P\left(\mu + ks > \mu + u_{\frac{1+p}{2}}\sigma\right) = P\left(\frac{s}{\sigma} > \frac{1}{k}u_{\frac{1+p}{2}}\right) = 1 - \alpha \qquad \text{(A.3)}$$

The distribution of s^2/σ^2 is $\chi^2/(n - 1)$ with $n - 1$ degrees of freedom, so it follows from the last equality in Formula (A.3) that

$$\frac{1}{k}u_{\frac{1+p}{2}} = \sqrt{\frac{\chi^2_\alpha(n-1)}{n-1}}$$

so

$$k = u_{\frac{1+p}{2}}\frac{\sqrt{n-1}}{\sqrt{\chi^2_\alpha(n-1)}}. \qquad \text{(A.4)}$$

Here $\chi^2_\alpha(n-1)$ is the α fractile of the χ^2 distribution with $n - 1$ degrees of freedom, so this is a value that is exceeded with probability $1 - \alpha$ by the random variable $s^2(n - 1)/\sigma^2$.

The variable k in Formula (A.4) is $k_2(n;p;1 - \alpha)$.

A.3 One-sided statistical tolerance interval with unknown mean and known standard deviation

Find k such that $\bar{x} + k\sigma$ satisfies that *at least* a proportion p of the population is below $\bar{x} + k\sigma$. Note that $\mu + u_p\sigma$ is the population tolerance limit in the sense that exactly a proportion p of the population is below that limit. So if

$$\bar{x} + k\sigma \geq \mu + u_p\sigma,$$

then the proportion of the population that is smaller than $\bar{x} + k\sigma$ is at least p. Thus the probability that a proportion of the population is at least p is $1 - \alpha$, if

$$P(\bar{x} + k\sigma \geq \mu + u_p\sigma) = 1 - \alpha. \qquad \text{(A.5)}$$

The probability on the left hand side of Formula (A.5) can be rewritten

$$P\left(\bar{x} + k\sigma \geq \mu + u_p\sigma\right) = P\left(\frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \geq \sqrt{n}u_p - \sqrt{n}k\right) = 1 - \alpha \qquad \text{(A.6)}$$

The variable $\frac{\sqrt{n}(\overline{x}-\mu)}{\sigma}$ in Formula (A.6) has a standard normal distribution, and it follows from the last equality in (A.6) that

$$\sqrt{n}u_p - \sqrt{n}k = u_\alpha,$$

which can be rewritten as

$$k = \frac{1}{\sqrt{n}}u_{1-\alpha} + u_p \qquad \text{(A.7)}$$

The variable k in Formula (A.7) is $k_3(n;p;1-\alpha)$.

The derivation was based on an upper tolerance interval, but a similar argument applies to the lower tolerance interval and $x_{\mathrm{L}} = \overline{x} - k_3(n;p;1-\alpha)s$ is the lower limit of a one-sided lower tolerance interval.

A.4 Two-sided statistical tolerance interval with unknown mean and known standard deviation

The general exact solution for the k-factor is the k that satisfies the equation

$$P\left(\mu + u_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} - k\sigma \le X \le \mu + u_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} + k\sigma\right) = p, \qquad \text{(A.8)}$$

where X has an N(μ, σ) distribution. This can be rewritten to give an exact formula for k in terms of the quantile of a non-central χ^2 distribution with one degree of freedom.

But first an argument why k in Formula (A.8) is the solution. The probability that a sample average $\overline{x}$ is in the interval bounded by $\mu \pm u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$ is $1-\alpha$.

Thus the proportion of intervals that are bounded by $\overline{x} \pm k\sigma$ and have their centres inside the interval bounded by $\mu \pm u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$ is $1 - \alpha$.

Now determine k to satisfy Formula (A.8). Considering the N(μ, σ^2) distribution it is clear that all intervals bounded by $\overline{x} \pm k\sigma$ have a probability greater than or equal to p if and only if $\overline{x}$ is in the interval bounded by $\mu \pm u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$, but the probability of this event is $1 - \alpha$.

With $b = u_{1-\alpha/2}\frac{1}{\sqrt{n}}$ and U denoting an N(μ, σ^2) distributed random variable equation, Formula (A.8) can be rewritten

$$\begin{aligned}
&P\left(\mu + u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} - k\sigma \le X \le \mu + u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} + k\sigma\right) = \\
&P\left(-k\sigma \le X - \mu - u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \le k\sigma\right) = \\
&P\left(-k \le \frac{X-\mu}{\sigma} - u_{1-\alpha/2}\frac{1}{\sqrt{n}} \le k\right) = \\
&P([U-b]^2 \le k^2) = p.
\end{aligned} \qquad \text{(A.9)}$$

Here $[U-b]^2$ has a non-central χ^2 distribution with 1 degree of freedom and non-centrality parameter $b^2=(u_{1-\alpha/2}\frac{1}{\sqrt{n}})^2$, and from the last equality in Formula (A.9)

$$k^2=\chi_p^2\left(1,\left(u_{1-\alpha/2}\frac{1}{\sqrt{n}}\right)^2\right)$$

and

$$k=\sqrt{\chi_p^2\left(1,\left(u_{1-\alpha/2}\frac{1}{\sqrt{n}}\right)^2\right)}, \tag{A.10}$$

where $\chi_P^2(1,(b)^2)$ denotes the p fractile of the non-central χ^2 distribution with 1 degree of freedom and non-centrality parameter b^2.

The variable k in Formula (A.10) is $k_4(n;p;1-\alpha)$.

A.5 One-sided statistical tolerance interval with unknown mean and unknown standard deviation

The task is to find k such that $\bar{x}+ks$ satisfies that *at least* a proportion p of the population is below $\bar{x}+ks$. Observe that $\mu+u_p\sigma$ is the population tolerance limit in the sense that exactly a proportion p of the population is below that limit. Now if

$$\bar{x}+ks\geq\mu+u_p\sigma$$

then the fraction of the population that is smaller than $\bar{x}+ks$ is at least p.

Thus the probability that a proportion of the population is at least p is $1-\alpha$, if

$$P(\bar{x}+ks\geq\mu+u_p\sigma)=1-\alpha. \tag{A.11}$$

This probability can be rewritten:

$$\begin{aligned}P(\bar{x}+ks\geq\mu+u_p\sigma) &= P(\bar{x}-\mu-u_p\sigma\geq-ks) =\\ P(\sqrt{n}(\bar{x}-\mu)-\sqrt{n}u_p\sigma\geq-\sqrt{n}ks) &= P\left(\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}-\sqrt{n}u_p\geq-\sqrt{n}k\frac{s}{\sigma}\right) =\\ P\left(\frac{-\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}+\sqrt{n}u_p}{\frac{s}{\sigma}}\leq\sqrt{n}k\right) &= 1-\alpha\end{aligned} \tag{A.12}$$

Here,

$$\frac{-\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}+\sqrt{n}u_p}{\frac{s}{\sigma}}$$

has a non-central t-distribution with $n-1$ degrees of freedom and non-centrality parameter $\sqrt{n}u_p$ so it follows from the final equation among the equations in Formula (A.12) that $\sqrt{n}k=t_{1-\alpha}(\sqrt{n}u_p,n-1)$, and the exact formula for k is

$$k=\frac{1}{\sqrt{n}}t_{1-\alpha}(\sqrt{n}u_p,n-1). \tag{A.13}$$

The variable k in Formula (A.13) is $k_C(n;p;1-\alpha)$. The factor $k_C(n;p;1-\alpha)$ is given for α=0,90; 0,95; 0,99; 0,999 and p= 0,90 and 0,99. The tabulated values are exact to the given number of decimal places.

In case the estimate of the variance, s^2, used in the derivation has a χ^2 distribution with f degrees of freedom for example because the variance is estimated from several independent samples with a common variance the k-factor is

$$k(n;f;p;1-\alpha)=\frac{1}{\sqrt{n}}t_{1-\alpha}(\sqrt{n}u_p,f) \quad \text{(A.14)}$$

Annex B
(informative)

Forms for statistical tolerance intervals

Form A — One sided statistical tolerance interval (unknown variance)

Determination of a one-sided statistical tolerance interval with coverage p at confidence level $1 - \alpha$

a) One-sided interval “to the left”

b) One-sided interval “to the right”

Determined values:

c) proportion of the population selected for the tolerance interval: $p =$

d) chosen confidence level: $1 - \alpha =$

e) sample size: $n =$

Tabulated factor: $k_C(n;p;1 - \alpha) =$

This value can be read from the tables given in Annex C for a range of values of n, p and $1 - \alpha$.

Calculations:

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j = \qquad ; s = \sqrt{\frac{n\sum_{j=1}^{n} x_j^2 - \left(\sum_{j=1}^{n} x_j\right)^2}{n(n-1)}} =$$

$k_C(n;p;1 - \alpha) \times s =$

Results:

f) One-sided interval “to the left”

The statistical tolerance interval with coverage p at confidence level $1 - \alpha$ has upper limit

$$x_U = \bar{x} + k_C(n;p;1-\alpha) \times s =$$

g) One-sided interval “to the right”

The statistical tolerance interval with coverage p at confidence level $1 - \alpha$ has lower limit

$$x_L = \bar{x} - k_C(n;p;1-\alpha) \times s =$$

Form B — Two-sided statistical tolerance interval (unknown variance)

Determination of a two-sided statistical tolerance interval with coverage p at confidence level $1 - \alpha$

Determined values:

h) proportion of the population selected for the statistical tolerance interval: p =

i) chosen confidence level: $1 - \alpha$ =

j) sample size: n =

Tabulated factor: $k_D(n;1;p;1 - \alpha)$ =

This value can be read from the first column of the tables given in Annex D for a range of values of n, p and $1 - \alpha$.

Calculations:

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j =$$

$$s = \sqrt{\frac{n\sum_{j=1}^{n} x_j^2 - \left(\sum_{j=1}^{n} x_j\right)^2}{n(n-1)}} =$$

$k_D(n;1;p;1 - \alpha) \times s$ =

Results:

The two-sided statistical tolerance interval with coverage p at confidence level $1 - \alpha$ has limits

$$x_L = \bar{x} - k_D(n;1;p;1-\alpha) \times s =$$

$$x_U = \bar{x} + k_D(n;1;p;1-\alpha) \times s =$$

Form C — Two-sided statistical tolerance intervals (unknown common variance)

Determination of a two-sided statistical tolerance interval with coverage p at confidence level $1 - \alpha$
Determined values:
k) proportion of the populations selected for the statistical tolerance intervals: p =
l) chosen confidence level: $1 - \alpha$ =
q) samples size: n =
r) number of samples: m =
Tabulated factor: $k_D(n;m;p;1 - \alpha)$ =
This value can be read from the tables given in Annex D for a range of values of n, m, p and $1 - \alpha$.
Calculations:
$\bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij} =$
$s_P = \sqrt{\frac{1}{m(n-1)}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij} - \bar{x}_i)^2} = \sqrt{\frac{1}{m}\sum_{i=1}^{m} s_i^2} =$
$k_D(n;m;p;1 - \alpha) \times s_P$ =
Results:
The two-sided statistical tolerance intervals with coverage p at confidence level $1 - \alpha$ has limits
$x_{Li} = \bar{x}_i - k_D(n;m;p;1-\alpha) \times s_P =$
$x_{Ui} = \bar{x}_i + k_D(n;m;p;1-\alpha) \times s_P =$
($i = 1,2,...,m$; $m \geq 2$)

Form D — Statistical tolerance interval for any distribution

Determination of a one- or two-sided distribution-free statistical tolerance interval with content p at confidence level $1 - \alpha$

a) One-sided upper interval $(-\infty, x_{(n-w+1)}]$

b) One-sided lower interval $[x_{(v)}, +\infty)$

c) Two-sided interval $[x_{(v)}, x_{(n-w+1)}]$

Specified values:

d) proportion of the population selected for the statistical tolerance interval: p = ________

e) chosen confidence level: $1 - \alpha$ = ________

f) vth smallest value of x to be used: v = ________

g) wth largest value of x to be used: w = ________

Note: Specify v as 0 for a one-sided upper interval or w as 0 for a one-sided lower interval.

Tabulated value: Sample size n for given p, $1 - \alpha$, and $v + w$:

This value can be read from Tables in Annex E for a range of values of p, $1 - \alpha$, and $v + w$.

Results:

The ____-sided statistical tolerance interval with content p = _____ at confidence level $1 - \alpha$ = _____ has a:

lower limit: $x_{(v)} = x_{(\_\_\_\_\_\_\_)}$ = __________

and

upper limit: $x_{(n-w+1)} = x_{(\_\_\_\_\_\_\_)}$ = __________

Annex C
(normative)

One-sided statistical tolerance limit factors, $k_C(n; p; 1-\alpha)$, for unknown σ

See Tables C.1 to C.4.

Table C.1 — Confidence level 90,0 %

(1 − α = 0,90)			
n	p		
	0,90	0,95	0,99
2	10,2528	13,0898	18,5001
3	4,2582	5,3115	7,3405
4	3,1879	3,9566	5,4383
5	2,7424	3,3999	4,6660
6	2,4937	3,0919	4,2426
7	2,3327	2,8938	3,9721
8	2,2186	2,7543	3,7826
9	2,1329	2,6500	3,6415
10	2,0657	2,5684	3,5317
11	2,0113	2,5027	3,4435
12	1,9662	2,4483	3,3707
13	1,9281	2,4025	3,3095
14	1,8954	2,3632	3,2572
15	1,8669	2,3290	3,2119
16	1,8418	2,2990	3,1721
17	1,8195	2,2725	3,1369
18	1,7996	2,2487	3,1055
19	1,7816	2,2273	3,0772
20	1,7653	2,2078	3,0516
22	1,7367	2,1739	3,0069
24	1,7124	2,1452	2,9692
26	1,6915	2,1204	2,9368
28	1,6732	2,0989	2,9086
30	1,6571	2,0799	2,8838
35	1,6239	2,0408	2,8329
40	1,5979	2,0103	2,7932
45	1,5769	1,9857	2,7613
50	1,5595	1,9653	2,7349
60	1,5321	1,9333	2,6936
70	1,5113	1,9091	2,6623
80	1,4948	1,8899	2,6377
90	1,4813	1,8743	2,6177
100	1,4701	1,8613	2,6010

Table C.1 *(continued)*

(1 − *α* = 0,90)			
n	*p*		
	0,90	0,95	0,99
150	1,4329	1,8182	2,5459
200	1,4113	1,7934	2,5141
250	1,3969	1,7767	2,4930
300	1,3863	1,7646	2,4775
400	1,3717	1,7478	2,4562
500	1,3618	1,7365	2,4418
1 000	1,3377	1,7089	2,4069
2 000	1,3210	1,6897	2,3828
5 000	1,3063	1,6731	2,3618
10 000	1,2990	1,6647	2,3513
20 000	1,2939	1,6589	2,3440
∞	1,2816	1,6449	2.3264

Table C.2 — Confidence level 95,0 %

(1 − *α* = 0,95)			
n	*p*		
	0,90	0,95	0,99
2	20,5815	26,2597	37,0936
3	6,1553	7,6560	10,5528
4	4,1620	5,1439	7,0424
5	3,4067	4,2027	5,7411
6	3,0063	3,7077	5,0620
7	2,7555	3,3995	4,6418
8	2,5820	3,1873	4,3539
9	2,4538	3,0313	4,1431
10	2,3547	2,9110	3,9812
11	2,2754	2,8150	3,8524
12	2,2102	2,7364	3,7471
13	2,1555	2,6706	3,6592
14	2,1088	2,6145	3,5846
15	2,0684	2,5661	3,5202
16	2,0330	2,5237	3,4640
17	2,0018	2,4863	3,4145
18	1,9738	2,4530	3,3704
19	1,9487	2,4231	3,3309
20	1,9260	2,3961	3,2952
22	1,8865	2,3490	3,2332
24	1,8530	2,3093	3,1811
26	1,8243	2,2754	3,1365
28	1,7993	2,2458	3,0979
30	1,7774	2,2199	3,0640
35	1,7323	2,1668	2,9946
40	1,6972	2,1255	2,9410

Table C.2 *(continued)*

(1 − *α* = 0,95)			
n	*p*		
	0,90	0,95	0,99
45	1,6690	2,0924	2,8980
50	1,6456	2,0650	2,8625
60	1,6090	2,0222	2,8071
70	1,5813	1,9899	2,7654
80	1,5594	1,9645	2,7327
90	1,5416	1,9438	2,7061
100	1,5268	1,9266	2,6840
150	1,4778	1,8699	2,6114
200	1,4496	1,8373	2,5698
250	1,4307	1,8155	2,5421
300	1,4170	1,7997	2,5219
400	1,3979	1,7778	2,4941
500	1,3851	1,7631	2,4755
1 000	1,3539	1,7273	2,4302
2 000	1,3323	1,7026	2,3990
5 000	1,3134	1,6811	2,3719
10 000	1,3040	1,6704	2,3584
20 000	1,2974	1,6629	2,3490
∞	1,2816	1,6449	2,3264

Table C.3 — Confidence level 99,0 %

(1 − *α* = 0,99)			
n	*P*		
	0,90	0,95	0,99
2	103,0287	131,4263	185,6170
3	13,9955	17,3702	23,8956
4	7,3799	9,0835	12,3873
5	5,3618	6,5784	8,9391
6	4,4111	5,4056	7,3346
7	3,8592	4,7279	6,4120
8	3,4973	4,2853	5,8118
9	3,2405	3,9723	5,3889
10	3,0480	3,7384	5,0738
11	2,8977	3,5562	4,8291
12	2,7768	3,4100	4,6331
13	2,6770	3,2896	4,4721
14	2,5932	3,1886	4,3372
15	2,5215	3,1024	4,2224
16	2,4595	3,0279	4,1233
17	2,4051	2,9628	4,0367
18	2,3571	2,9052	3,9604
19	2,3142	2,8539	3,8925
20	2,2757	2,8079	3,8316

Table C.3 *(continued)*

(1 − α = 0,99)			
n	P		
	0,90	0,95	0,99
22	2,2092	2,7286	3,7268
24	2,1536	2,6624	3,6396
26	2,1063	2,6062	3,5656
28	2,0655	2,5578	3,5020
30	2,0299	2,5155	3,4466
35	1,9575	2,4299	3,3344
40	1,9018	2,3642	3,2486
45	1,8573	2,3118	3,1804
50	1,8208	2,2689	3,1247
60	1,7641	2,2024	3,0383
70	1,7216	2,1527	2,9740
80	1,6883	2,1138	2,9238
90	1,6614	2,0824	2,8832
100	1,6390	2,0563	2,8497
150	1,5658	1,9713	2,7405
200	1,5241	1,9230	2,6787
250	1,4963	1,8909	2,6377
300	1,4762	1,8676	2,6081
400	1,4484	1,8357	2,5674
500	1,4298	1,8143	2,5402
1 000	1,3847	1,7625	2,4746
2 000	1,3537	1,7270	2,4298
5 000	1,3267	1,6963	2,3910
10 000	1,3134	1,6810	2,3718
20 000	1,3040	1,6704	2,3584
∞	1,2816	1,6449	2,3264

Table C.4 — Confidence level 99,9 %

(1 − α = 0,999)			
n	p		
	0,90	0,95	0,99
2	1030,3362	1314,3157	1856,2311
3	44,4199	55,1055	75,7741
4	16,1217	19,8127	26,9791
5	9,7816	11,9695	16,2230
6	7,2465	8,8486	11,9645
7	5,9206	7,2223	9,7538
8	5,1127	6,2344	8,4151
9	4,5700	5,5725	7,5206
10	4,1801	5,0981	6,8810
11	3,8860	4,7410	6,4006
12	3,6558	4,4621	6,0261
13	3,4705	4,2378	5,7255

Table C.4 *(continued)*

(1 − α = 0,999)			
n	p		
	0,90	0,95	0,99
14	3,3177	4,0532	5,4786
15	3,1894	3,8984	5,2718
16	3,0800	3,7666	5,0960
17	2,9854	3,6528	4,9444
18	2,9027	3,5535	4,8122
19	2,8298	3,4659	4,6958
20	2,7649	3,3881	4,5925
22	2,6542	3,2555	4,4167
24	2,5630	3,1465	4,2725
26	2,4864	3,0551	4,1518
28	2,4210	2,9772	4,0490
30	2,3644	2,9098	3,9602
35	2,2509	2,7750	3,7829
40	2,1650	2,6732	3,6494
45	2,0973	2,5931	3,5447
50	2,0422	2,5281	3,4598
60	1,9576	2,4283	3,3299
70	1,8950	2,3548	3,2343
80	1,8464	2,2978	3,1604
90	1,8073	2,2520	3,1012
100	1,7750	2,2143	3,0524
150	1,6707	2,0927	2,8957
200	1,6120	2,0245	2,8082
250	1,5732	1,9796	2,7507
300	1,5453	1,9473	2,7094
400	1,5070	1,9031	2,6530
500	1,4814	1,8736	2,6155
1 000	1,4199	1,8029	2,5257
2 000	1,3780	1,7549	2,4649
5 000	1,3418	1,7135	2,4127
10 000	1,3239	1,6931	2,3870
20 000	1,3114	1,6788	2,3690
∞	1,2816	1,6449	2,3264

Annex D
(normative)

Two-sided statistical tolerance limit factors, $k_D(n; m; p; 1-\alpha)$, for unknown common σ (m samples)

See Tables D.1 to D.12.

Table D.1 — Confidence level 90,0 % and proportion 90,0 % ($1 - \alpha = 0{,}90$; $p = 0{,}90$)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	15,5124	6,0755	4,5088	3,8875	3,5544	3,3461	3,2032	3,0989	3,0193	2,9565
3	5,7881	3,6819	3,1564	2,9142	2,7733	2,6805	2,6146	2,5652	2,5268	2,4961
4	4,1571	3,0537	2,7366	2,5822	2,4894	2,4272	2,3823	2,3483	2,3216	2,3001
5	3,4993	2,7522	2,5209	2,4046	2,3336	2,2853	2,2502	2,2234	2,2023	2,1852
6	3,1406	2,5712	2,3863	2,2915	2,2329	2,1927	2,1632	2,1406	2,1227	2,1082
7	2,9128	2,4489	2,2932	2,2121	2,1616	2,1266	2,1009	2,0812	2,0654	2,0526
8	2,7542	2,3600	2,2244	2,1530	2,1081	2,0769	2,0539	2,0361	2,0220	2,0104
9	2,6368	2,2921	2,1712	2,1069	2,0663	2,0380	2,0170	2,0008	1,9878	1,9771
10	2,5460	2,2384	2,1287	2,0700	2,0327	2,0066	1,9872	1,9722	1,9601	1,9502
11	2,4734	2,1946	2,0938	2,0396	2,0050	1,9807	1,9626	1,9485	1,9372	1,9279
12	2,4140	2,1581	2,0646	2,0141	1,9817	1,9589	1,9419	1,9286	1,9180	1,9092
13	2,3643	2,1273	2,0398	1,9923	1,9618	1,9403	1,9242	1,9116	1,9015	1,8931
14	2,3220	2,1008	2,0184	1,9735	1,9446	1,9242	1,9089	1,8969	1,8872	1,8793
15	2,2855	2,0777	1,9998	1,9571	1,9296	1,9101	1,8955	1,8840	1,8748	1,8671
16	2,2537	2,0574	1,9833	1,9426	1,9163	1,8977	1,8837	1,8727	1,8638	1,8564
17	2,2257	2,0394	1,9687	1,9298	1,9045	1,8866	1,8731	1,8626	1,8540	1,8469
18	2,2008	2,0233	1,9556	1,9182	1,8940	1,8767	1,8637	1,8535	1,8452	1,8384
19	2,1785	2,0089	1,9438	1,9078	1,8844	1,8678	1,8552	1,8453	1,8373	1,8307
20	2,1584	1,9958	1,9331	1,8984	1,8758	1,8596	1,8475	1,8379	1,8302	1,8237
22	2,1235	1,9729	1,9144	1,8819	1,8606	1,8455	1,8340	1,8250	1,8176	1,8115
24	2,0943	1,9536	1,8986	1,8679	1,8478	1,8335	1,8226	1,8140	1,8070	1,8013
26	2,0693	1,9371	1,8851	1,8559	1,8368	1,8232	1,8128	1,8046	1,7980	1,7924
28	2,0478	1,9227	1,8733	1,8455	1,8273	1,8142	1,8043	1,7965	1,7901	1,7848
30	2,0289	1,9101	1,8629	1,8363	1,8189	1,8063	1,7968	1,7893	1,7832	1,7780
35	1,9906	1,8843	1,8417	1,8176	1,8017	1,7902	1,7815	1,7747	1,7690	1,7643
40	1,9611	1,8643	1,8252	1,8030	1,7884	1,7778	1,7697	1,7634	1,7581	1,7538
45	1,9376	1,8483	1,8121	1,7914	1,7777	1,7679	1,7603	1,7543	1,7494	1,7454
50	1,9184	1,8352	1,8012	1,7818	1,7690	1,7597	1,7526	1,7469	1,7423	1,7385
60	1,8885	1,8147	1,7844	1,7670	1,7554	1,7470	1,7406	1,7355	1,7313	1,7278
70	1,8662	1,7994	1,7718	1,7558	1,7452	1,7375	1,7316	1,7269	1,7231	1,7199
80	1,8489	1,7874	1,7619	1,7471	1,7373	1,7301	1,7247	1,7203	1,7167	1,7137
90	1,8348	1,7778	1,7539	1,7401	1,7309	1,7242	1,7190	1,7149	1,7116	1,7087
100	1,8232	1,7697	1,7473	1,7343	1,7256	1,7193	1,7144	1,7105	1,7073	1,7047
150	1,7856	1,7436	1,7257	1,7154	1,7084	1,7033	1,6994	1,6963	1,6937	1,6915

Table D.1 *(continued)*

n	m									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
200	1,7643	1,7287	1,7136	1,7047	1,6987	1,6943	1,6910	1,6883	1,6861	1,6842
250	1,7502	1,7189	1,7055	1,6976	1,6923	1,6884	1,6854	1,6830	1,6811	1,6794
300	1,7401	1,7118	1,6997	1,6925	1,6877	1,6842	1,6815	1,6793	1,6775	1,6760
400	1,7262	1,7021	1,6917	1,6856	1,6814	1,6784	1,6761	1,6742	1,6726	1,6713
500	1,7169	1,6956	1,6864	1,6809	1,6773	1,6746	1,6725	1,6708	1,6694	1,6682
1 000	1,6947	1,6800	1,6736	1,6698	1,6672	1,6653	1,6639	1,6627	1,6617	1,6609
2 000	1,6795	1,6693	1,6649	1,6622	1,6604	1,6591	1,6581	1,6572	1,6565	1,6560
5 000	1,6665	1,6601	1,6574	1,6557	1,6546	1,6537	1,6531	1,6526	1,6521	1,6518
10 000	1,6601	1,6556	1,6536	1,6525	1,6517	1,6511	1,6506	1,6503	1,6500	1,6497
20 000	1,6556	1,6524	1,6511	1,6502	1,6497	1,6493	1,6489	1,6487	1,6485	1,6483
∞	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449

Table D.2 — Confidence level 90,0 % and proportion 95,0 % ($1 - \alpha = 0,90$; $p = 0,95$)

n	m									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	18,2208	7,1197	5,2743	4,5412	4,1473	3,9005	3,7308	3,6067	3,5117	3,4367
3	6,8233	4,3320	3,7087	3,4207	3,2528	3,1420	3,0630	3,0038	2,9575	2,9205
4	4,9127	3,6034	3,2262	3,0419	2,9311	2,8566	2,8027	2,7618	2,7297	2,7037
5	4,1425	3,2544	2,9787	2,8400	2,7551	2,6972	2,6551	2,6229	2,5974	2,5768
6	3,7226	3,0449	2,8245	2,7112	2,6411	2,5930	2,5577	2,5306	2,5091	2,4916
7	3,4558	2,9034	2,7176	2,6208	2,5604	2,5186	2,4878	2,4641	2,4452	2,4298
8	3,2699	2,8004	2,6385	2,5532	2,4996	2,4624	2,4348	2,4136	2,3966	2,3827
9	3,1323	2,7216	2,5773	2,5006	2,4521	2,4182	2,3931	2,3737	2,3581	2,3454
10	3,0258	2,6591	2,5282	2,4582	2,4137	2,3825	2,3593	2,3413	2,3269	2,3150
11	2,9406	2,6082	2,4880	2,4232	2,3819	2,3529	2,3313	2,3145	2,3010	2,2899
12	2,8707	2,5658	2,4542	2,3938	2,3552	2,3280	2,3077	2,2918	2,2791	2,2686
13	2,8123	2,5298	2,4254	2,3687	2,3323	2,3066	2,2874	2,2724	2,2603	2,2503
14	2,7625	2,4988	2,4006	2,3470	2,3125	2,2881	2,2699	2,2556	2,2440	2,2345
15	2,7196	2,4718	2,3789	2,3280	2,2951	2,2719	2,2545	2,2408	2,2298	2,2206
16	2,6821	2,4481	2,3597	2,3112	2,2798	2,2576	2,2408	2,2277	2,2171	2,2084
17	2,6491	2,4270	2,3427	2,2962	2,2661	2,2448	2,2287	2,2161	2,2059	2,1974
18	2,6197	2,4082	2,3274	2,2828	2,2539	2,2333	2,2178	2,2056	2,1958	2,1876
19	2,5934	2,3912	2,3136	2,2707	2,2428	2,2229	2,2079	2,1962	2,1866	2,1787
20	2,5697	2,3758	2,3011	2,2597	2,2327	2,2135	2,1990	2,1876	2,1783	2,1706
22	2,5285	2,3490	2,2793	2,2404	2,2151	2,1970	2,1833	2,1725	2,1638	2,1565
24	2,4940	2,3263	2,2607	2,2241	2,2001	2,1830	2,1700	2,1598	2,1515	2,1446
26	2,4645	2,3068	2,2448	2,2100	2,1873	2,1710	2,1586	2,1489	2,1409	2,1343
28	2,4390	2,2898	2,2309	2,1978	2,1761	2,1605	2,1487	2,1393	2,1317	2,1254
30	2,4166	2,2749	2,2187	2,1870	2,1662	2,1513	2,1399	2,1309	2,1236	2,1175
35	2,3712	2,2445	2,1937	2,1649	2,1460	2,1324	2,1220	2,1138	2,1071	2,1015
40	2,3363	2,2209	2,1743	2,1478	2,1303	2,1177	2,1081	2,1005	2,0943	2,0891
45	2,3084	2,2020	2,1587	2,1341	2,1178	2,1060	2,0970	2,0899	2,0841	2,0792
50	2,2855	2,1864	2,1459	2,1228	2,1075	2,0964	2,0879	2,0812	2,0757	2,0711
60	2,2500	2,1621	2,1260	2,1052	2,0914	2,0814	2,0737	2,0677	2,0627	2,0585

Table D.2 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
70	2,2236	2,1440	2,1110	2,0920	2,0794	2,0702	2,0632	2,0576	2,0530	2,0491
80	2,2029	2,1297	2,0993	2,0817	2,0699	2,0614	2,0549	2,0497	2,0454	2,0418
90	2,1862	2,1182	2,0898	2,0733	2,0624	2,0544	2,0482	2,0433	2,0393	2,0360
100	2,1724	2,1087	2,0819	2,0664	2,0561	2,0485	2,0427	2,0381	2,0343	2,0311
150	2,1276	2,0775	2,0563	2,0439	2,0356	2,0296	2,0249	2,0212	2,0181	2,0155
200	2,1022	2,0599	2,0418	2,0312	2,0241	2,0189	2,0149	2,0117	2,0090	2,0068
250	2,0855	2,0482	2,0322	2,0228	2,0165	2,0119	2,0083	2,0055	2,0031	2,0011
300	2,0734	2,0397	2,0253	2,0168	2,0110	2,0068	2,0036	2,0010	1,9988	1,9970
400	2,0569	2,0282	2,0158	2,0085	2,0035	1,9999	1,9971	1,9949	1,9930	1,9915
500	2,0458	2,0204	2,0094	2,0029	1,9986	1,9953	1,9928	1,9908	1,9892	1,9878
1 000	2,0193	2,0018	1,9942	1,9897	1,9866	1,9844	1,9826	1,9812	1,9800	1,9791
2 000	2,0013	1,9891	1,9838	1,9806	1,9785	1,9769	1,9757	1,9747	1,9739	1,9732
5 000	1,9857	1,9782	1,9749	1,9729	1,9715	1,9705	1,9698	1,9691	1,9686	1,9682
10 000	1,9781	1,9728	1,9704	1,9690	1,9681	1,9674	1,9669	1,9664	1,9661	1,9658
20 000	1,9727	1,9690	1,9673	1,9664	1,9657	1,9652	1,9648	1,9645	1,9643	1,9640
∞	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600

Table D.3 — Confidence level 90,0 % and proportion 99,0 % ($1 - \alpha = 0{,}90$; $p = 0{,}99$)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	23,4235	9,1259	6,7452	5,7970	5,2861	4,9651	4,7436	4,5811	4,4565	4,3577
3	8,8187	5,5844	4,7723	4,3955	4,1749	4,0287	3,9242	3,8454	3,7837	3,7341
4	6,3722	4,6643	4,1701	3,9277	3,7814	3,6825	3,6108	3,5562	3,5131	3,4782
5	5,3868	4,2250	3,8628	3,6798	3,5674	3,4906	3,4344	3,3914	3,3573	3,3295
6	4,8498	3,9616	3,6715	3,5220	3,4291	3,3652	3,3182	3,2820	3,2532	3,2297
7	4,5085	3,7836	3,5389	3,4111	3,3311	3,2756	3,2347	3,2030	3,1778	3,1572
8	4,2707	3,6541	3,4408	3,3281	3,2572	3,2078	3,1712	3,1428	3,1202	3,1016
9	4,0945	3,5549	3,3646	3,2633	3,1991	3,1543	3,1210	3,0951	3,0744	3,0574
10	3,9580	3,4761	3,3035	3,2110	3,1521	3,1109	3,0802	3,0563	3,0371	3,0213
11	3,8488	3,4117	3,2533	3,1678	3,1132	3,0748	3,0462	3,0239	3,0059	2,9912
12	3,7591	3,3581	3,2110	3,1313	3,0803	3,0443	3,0174	2,9964	2,9795	2,9656
13	3,6840	3,3125	3,1750	3,1001	3,0520	3,0181	2,9927	2,9728	2,9568	2,9436
14	3,6201	3,2732	3,1438	3,0731	3,0275	2,9953	2,9711	2,9522	2,9370	2,9244
15	3,5649	3,2389	3,1165	3,0493	3,0060	2,9753	2,9522	2,9341	2,9196	2,9075
16	3,5166	3,2087	3,0923	3,0283	2,9869	2,9575	2,9354	2,9181	2,9041	2,8925
17	3,4741	3,1819	3,0708	3,0095	2,9698	2,9416	2,9204	2,9037	2,8902	2,8791
18	3,4362	3,1579	3,0515	2,9926	2,9545	2,9273	2,9069	2,8908	2,8778	2,8670
19	3,4022	3,1362	3,0340	2,9774	2,9406	2,9144	2,8946	2,8791	2,8665	2,8560
20	3,3716	3,1165	3,0181	2,9635	2,9279	2,9026	2,8835	2,8684	2,8562	2,8461
22	3,3183	3,0822	2,9903	2,9391	2,9057	2,8819	2,8639	2,8497	2,8381	2,8286
24	3,2736	3,0530	2,9667	2,9184	2,8869	2,8643	2,8472	2,8337	2,8228	2,8137
26	3,2354	3,0280	2,9464	2,9006	2,8706	2,8491	2,8328	2,8200	2,8095	2,8008
28	3,2023	3,0062	2,9286	2,8850	2,8564	2,8358	2,8203	2,8080	2,7980	2,7896
30	3,1734	2,9870	2,9130	2,8712	2,8438	2,8241	2,8092	2,7974	2,7878	2,7797

Table D.3 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
35	3,1143	2,9477	2,8808	2,8430	2,8180	2,8001	2,7864	2,7756	2,7668	2,7594
40	3,0688	2,9171	2,8558	2,8210	2,7980	2,7814	2,7687	2,7587	2,7505	2,7437
45	3,0325	2,8926	2,8357	2,8033	2,7818	2,7663	2,7545	2,7451	2,7375	2,7310
50	3,0027	2,8724	2,8191	2,7887	2,7685	2,7539	2,7428	2,7339	2,7267	2,7206
60	2,9564	2,8408	2,7932	2,7659	2,7477	2,7346	2,7245	2,7165	2,7099	2,7045
70	2,9218	2,8171	2,7737	2,7488	2,7321	2,7201	2,7108	2,7035	2,6974	2,6924
80	2,8947	2,7985	2,7585	2,7353	2,7199	2,7087	2,7001	2,6932	2,6876	2,6829
90	2,8729	2,7835	2,7461	2,7245	2,7100	2,6995	2,6914	2,6850	2,6797	2,6753
100	2,8548	2,7710	2,7358	2,7155	2,7018	2,6919	2,6843	2,6782	2,6732	2,6690
150	2,7960	2,7302	2,7023	2,6861	2,6751	2,6672	2,6610	2,6561	2,6521	2,6487
200	2,7627	2,7070	2,6833	2,6694	2,6600	2,6532	2,6479	2,6437	2,6402	2,6373
250	2,7407	2,6917	2,6707	2,6584	2,6501	2,6440	2,6393	2,6355	2,6324	2,6298
300	2,7249	2,6806	2,6616	2,6504	2,6429	2,6374	2,6331	2,6297	2,6269	2,6245
400	2,7031	2,6654	2,6491	2,6396	2,6331	2,6283	2,6246	2,6217	2,6193	2,6172
500	2,6886	2,6553	2,6408	2,6323	2,6265	2,6223	2,6190	2,6164	2,6142	2,6124
1 000	2,6538	2,6308	2,6208	2,6148	2,6108	2,6079	2,6056	2,6037	2,6022	2,6009
2 000	2,6301	2,6141	2,6071	2,6030	2,6002	2,5981	2,5965	2,5952	2,5941	2,5932
5 000	2,6097	2,5998	2,5954	2,5928	2,5910	2,5897	2,5887	2,5879	2,5872	2,5866
10 000	2,5996	2,5926	2,5896	2,5877	2,5865	2,5856	2,5849	2,5843	2,5838	2,5834
20 000	2,5926	2,5877	2,5855	2,5842	2,5834	2,5827	2,5822	2,5818	2,5815	2,5812
∞	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759

Table D.4 — Confidence level 95,0 % and proportion 90,0 % ($1 - \alpha = 0{,}95$; $p = 0{,}90$)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	31,0923	8,7252	5,8380	4,7912	4,2571	3,9341	3,7179	3,5630	3,4468	3,3565
3	8,3060	4,5251	3,6939	3,3300	3,1251	2,9934	2,9017	2,8341	2,7824	2,7416
4	5,3681	3,5647	3,0909	2,8693	2,7400	2,6550	2,5949	2,5502	2,5157	2,4883
5	4,2907	3,1276	2,7925	2,6300	2,5332	2,4688	2,4229	2,3885	2,3618	2,3405
6	3,7326	2,8726	2,6100	2,4796	2,4009	2,3480	2,3100	2,2814	2,2592	2,2414
7	3,3896	2,7033	2,4852	2,3750	2,3077	2,2623	2,2294	2,2046	2,1851	2,1696
8	3,1561	2,5818	2,3937	2,2974	2,2381	2,1978	2,1685	2,1463	2,1289	2,1149
9	2,9861	2,4899	2,3234	2,2372	2,1839	2,1474	2,1208	2,1005	2,0846	2,0717
10	2,8564	2,4175	2,2674	2,1891	2,1403	2,1067	2,0822	2,0634	2,0487	2,0367
11	2,7537	2,3589	2,2217	2,1495	2,1044	2,0732	2,0503	2,0328	2,0190	2,0077
12	2,6703	2,3104	2,1835	2,1164	2,0742	2,0450	2,0235	2,0070	1,9939	1,9833
13	2,6011	2,2694	2,1512	2,0883	2,0485	2,0210	2,0006	1,9850	1,9726	1,9625
14	2,5425	2,2343	2,1233	2,0640	2,0264	2,0002	1,9809	1,9659	1,9541	1,9444
15	2,4922	2,2039	2,0991	2,0428	2,0070	1,9821	1,9636	1,9493	1,9379	1,9286
16	2,4486	2,1771	2,0777	2,0241	1,9899	1,9661	1,9483	1,9346	1,9237	1,9147
17	2,4103	2,1535	2,0588	2,0075	1,9748	1,9518	1,9348	1,9215	1,9110	1,9023
18	2,3764	2,1324	2,0418	1,9926	1,9612	1,9391	1,9226	1,9099	1,8996	1,8913
19	2,3461	2,1135	2,0266	1,9793	1,9489	1,9276	1,9117	1,8993	1,8894	1,8813
20	2,3188	2,0963	2,0128	1,9671	1,9378	1,9172	1,9017	1,8898	1,8801	1,8722

Table D.4 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
22	2,2718	2,0665	1,9887	1,9460	1,9184	1,8990	1,8844	1,8731	1,8640	1,8565
24	2,2325	2,0414	1,9683	1,9281	1,9020	1,8836	1,8698	1,8590	1,8503	1,8432
26	2,1991	2,0199	1,9509	1,9127	1,8880	1,8704	1,8573	1,8470	1,8386	1,8318
28	2,1703	2,0012	1,9357	1,8994	1,8758	1,8590	1,8464	1,8365	1,8285	1,8219
30	2,1452	1,9849	1,9225	1,8877	1,8651	1,8490	1,8369	1,8273	1,8197	1,8133
35	2,0943	1,9515	1,8953	1,8638	1,8432	1,8285	1,8174	1,8087	1,8016	1,7957
40	2,0553	1,9258	1,8743	1,8453	1,8263	1,8127	1,8024	1,7943	1,7877	1,7822
45	2,0244	1,9052	1,8575	1,8306	1,8128	1,8001	1,7905	1,7828	1,7767	1,7715
50	1,9991	1,8883	1,8437	1,8184	1,8018	1,7898	1,7807	1,7735	1,7676	1,7627
60	1,9599	1,8621	1,8223	1,7996	1,7846	1,7738	1,7655	1,7590	1,7537	1,7492
70	1,9308	1,8425	1,8062	1,7855	1,7717	1,7618	1,7542	1,7482	1,7433	1,7392
80	1,9082	1,8271	1,7937	1,7745	1,7617	1,7525	1,7455	1,7399	1,7353	1,7314
90	1,8899	1,8147	1,7835	1,7656	1,7537	1,7450	1,7384	1,7331	1,7288	1,7252
100	1,8749	1,8044	1,7752	1,7583	1,7470	1,7388	1,7326	1,7276	1,7235	1,7201
150	1,8260	1,7710	1,7478	1,7344	1,7254	1,7188	1,7137	1,7097	1,7064	1,7036
200	1,7985	1,7521	1,7324	1,7209	1,7132	1,7075	1,7032	1,6997	1,6968	1,6944
250	1,7803	1,7395	1,7221	1,7120	1,7051	1,7001	1,6962	1,6931	1,6906	1,6884
300	1,7673	1,7305	1,7148	1,7055	1,6993	1,6948	1,6912	1,6884	1,6861	1,6842
400	1,7494	1,7181	1,7046	1,6967	1,6914	1,6875	1,6844	1,6820	1,6800	1,6783
500	1,7374	1,7098	1,6979	1,6908	1,6861	1,6826	1,6799	1,6777	1,6760	1,6744
1 000	1,7088	1,6898	1,6816	1,6767	1,6734	1,6709	1,6690	1,6675	1,6663	1,6652
2 000	1,6894	1,6762	1,6705	1,6670	1,6647	1,6630	1,6617	1,6606	1,6598	1,6590
5 000	1,6726	1,6645	1,6609	1,6587	1,6573	1,6562	1,6554	1,6547	1,6542	1,6537
10 000	1,6644	1,6586	1,6561	1,6546	1,6536	1,6528	1,6523	1,6518	1,6514	1,6511
20 000	1,6586	1,6546	1,6528	1,6517	1,6510	1,6505	1,6501	1,6497	1,6495	1,6492
∞	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449

Table D.5 — Confidence level 95,0 % and proportion 95,0 % ($1 - \alpha = 0{,}95$; $p = 0{,}95$)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	36,5193	10,2199	6,8215	5,5868	4,9552	4,5720	4,3146	4,1298	3,9907	3,8821
3	9,7888	5,3184	4,3321	3,8987	3,6535	3,4952	3,3844	3,3025	3,2395	3,1895
4	6,3411	4,2013	3,6366	3,3713	3,2157	3,1130	3,0401	2,9855	2,9432	2,9095
5	5,0769	3,6939	3,2936	3,0986	2,9820	2,9041	2,8482	2,8062	2,7734	2,7472
6	4,4222	3,3981	3,0841	2,9276	2,8327	2,7687	2,7225	2,6876	2,6603	2,6384
7	4,0196	3,2018	2,9408	2,8085	2,7275	2,6725	2,6326	2,6024	2,5786	2,5595
8	3,7456	3,0609	2,8357	2,7201	2,6488	2,6001	2,5646	2,5376	2,5163	2,4992
9	3,5459	2,9541	2,7548	2,6515	2,5873	2,5433	2,5111	2,4865	2,4671	2,4514
10	3,3935	2,8700	2,6904	2,5964	2,5377	2,4973	2,4677	2,4450	2,4271	2,4125
11	3,2728	2,8018	2,6376	2,5511	2,4969	2,4594	2,4318	2,4106	2,3938	2,3802
12	3,1747	2,7452	2,5936	2,5131	2,4625	2,4273	2,4015	2,3815	2,3657	2,3528
13	3,0932	2,6975	2,5561	2,4807	2,4331	2,4000	2,3755	2,3566	2,3416	2,3294
14	3,0242	2,6565	2,5238	2,4527	2,4077	2,3763	2,3530	2,3350	2,3207	2,3090
15	2,9650	2,6209	2,4957	2,4283	2,3854	2,3555	2,3333	2,3161	2,3024	2,2912

Table D.5 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
16	2,9135	2,5897	2,4709	2,4067	2,3658	2,3371	2,3158	2,2993	2,2862	2,2754
17	2,8684	2,5620	2,4488	2,3875	2,3483	2,3208	2,3003	2,2844	2,2717	2,2613
18	2,8283	2,5373	2,4291	2,3702	2,3326	2,3061	2,2864	2,2710	2,2587	2,2487
19	2,7926	2,5151	2,4113	2,3547	2,3184	2,2928	2,2738	2,2589	2,2470	2,2373
20	2,7604	2,4950	2,3952	2,3406	2,3055	2,2808	2,2623	2,2479	2,2364	2,2269
22	2,7048	2,4599	2,3670	2,3160	2,2830	2,2598	2,2423	2,2287	2,2178	2,2088
24	2,6583	2,4304	2,3432	2,2951	2,2640	2,2419	2,2254	2,2125	2,2021	2,1935
26	2,6188	2,4051	2,3227	2,2771	2,2476	2,2266	2,2108	2,1985	2,1886	2,1803
28	2,5847	2,3831	2,3049	2,2615	2,2333	2,2133	2,1982	2,1864	2,1768	2,1689
30	2,5549	2,3638	2,2893	2,2478	2,2208	2,2016	2,1871	2,1757	2,1665	2,1589
35	2,4946	2,3244	2,2573	2,2197	2,1952	2,1776	2,1643	2,1539	2,1455	2,1384
40	2,4484	2,2940	2,2326	2,1980	2,1753	2,1591	2,1468	2,1371	2,1292	2,1227
45	2,4117	2,2696	2,2128	2,1806	2,1594	2,1443	2,1327	2,1237	2,1163	2,1101
50	2,3816	2,2496	2,1964	2,1663	2,1464	2,1321	2,1212	2,1126	2,1056	2,0998
60	2,3351	2,2185	2,1710	2,1440	2,1261	2,1132	2,1033	2,0956	2,0892	2,0839
70	2,3005	2,1952	2,1520	2,1273	2,1109	2,0991	2,0900	2,0828	2,0770	2,0721
80	2,2736	2,1770	2,1371	2,1142	2,0990	2,0880	2,0796	2,0729	2,0675	2,0629
90	2,2519	2,1622	2,1251	2,1037	2,0895	2,0792	2,0713	2,0650	2,0598	2,0555
100	2,2339	2,1500	2,1151	2,0950	2,0815	2,0718	2,0643	2,0584	2,0535	2,0495
150	2,1758	2,1102	2,0826	2,0666	2,0558	2,0480	2,0420	2,0372	2,0332	2,0299
200	2,1430	2,0877	2,0642	2,0505	2,0413	2,0346	2,0294	2,0253	2,0219	2,0190
250	2,1214	2,0728	2,0520	2,0399	2,0317	2,0258	2,0212	2,0175	2,0144	2,0119
300	2,1058	2,0620	2,0432	2,0322	2,0248	2,0194	2,0152	2,0119	2,0091	2,0068
400	2,0845	2,0472	2,0312	2,0217	2,0154	2,0107	2,0071	2,0042	2,0018	1,9998
500	2,0703	2,0373	2,0231	2,0147	2,0091	2,0049	2,0017	1,9991	1,9970	1,9952
1 000	2,0362	2,0135	2,0037	1,9979	1,9939	1,9910	1,9888	1,9870	1,9855	1,9842
2 000	2,0130	1,9973	1,9905	1,9864	1,9836	1,9816	1,9800	1,9788	1,9777	1,9768
5 000	1,9930	1,9833	1,9790	1,9765	1,9748	1,9735	1,9725	1,9717	1,9710	1,9705
10 000	1,9832	1,9764	1,9734	1,9716	1,9704	1,9695	1,9688	1,9682	1,9677	1,9674
20 000	1,9763	1,9715	1,9694	1,9682	1,9673	1,9667	1,9662	1,9658	1,9655	1,9652
∞	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600

Table D.6 — Confidence level 95,0 % and proportion 99,0 % (1 − α = 0,95; p = 0,99)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	46,9445	13,0925	8,7128	7,1173	6,2983	5,7995	5,4632	5,2207	5,0372	4,8934
3	12,6472	6,8474	5,5623	4,9943	4,6711	4,4612	4,3133	4,2032	4,1180	4,0500
4	8,2207	5,4302	4,6896	4,3392	4,1324	3,9949	3,8965	3,8225	3,7647	3,7182
5	6,5980	4,7884	4,2614	4,0029	3,8472	3,7425	3,6668	3,6095	3,5645	3,5283
6	5,7578	4,4149	4,0005	3,7926	3,6657	3,5796	3,5170	3,4694	3,4320	3,4017
7	5,2411	4,1672	3,8223	3,6464	3,5381	3,4640	3,4100	3,3688	3,3362	3,3099
8	4,8893	3,9893	3,6916	3,5378	3,4424	3,3769	3,3290	3,2922	3,2632	3,2396
9	4,6329	3,8544	3,5909	3,4534	3,3677	3,3085	3,2651	3,2317	3,2052	3,1837
10	4,4370	3,7481	3,5105	3,3856	3,3073	3,2531	3,2131	3,1824	3,1580	3,1381

Table D.6 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
11	4,2818	3,6618	3,4447	3,3297	3,2573	3,2071	3,1700	3,1414	3,1186	3,1000
12	4,1556	3,5901	3,3896	3,2828	3,2152	3,1682	3,1334	3,1066	3,0852	3,0677
13	4,0506	3,5295	3,3426	3,2426	3,1791	3,1349	3,1021	3,0767	3,0564	3,0398
14	3,9617	3,4775	3,3021	3,2078	3,1478	3,1059	3,0747	3,0506	3,0313	3,0155
15	3,8853	3,4323	3,2667	3,1774	3,1204	3,0804	3,0507	3,0277	3,0093	2,9941
16	3,8189	3,3925	3,2355	3,1504	3,0960	3,0579	3,0295	3,0074	2,9897	2,9752
17	3,7606	3,3572	3,2077	3,1264	3,0743	3,0377	3,0104	2,9892	2,9722	2,9582
18	3,7089	3,3257	3,1828	3,1048	3,0548	3,0196	2,9933	2,9728	2,9564	2,9429
19	3,6626	3,2973	3,1603	3,0853	3,0372	3,0032	2,9778	2,9580	2,9421	2,9290
20	3,6210	3,2716	3,1398	3,0676	3,0211	2,9883	2,9637	2,9445	2,9291	2,9164
22	3,5491	3,2267	3,1041	3,0365	2,9929	2,9620	2,9389	2,9208	2,9062	2,8942
24	3,4888	3,1888	3,0737	3,0102	2,9690	2,9398	2,9178	2,9006	2,8868	2,8753
26	3,4375	3,1562	3,0476	2,9874	2,9483	2,9205	2,8996	2,8833	2,8700	2,8591
28	3,3933	3,1280	3,0249	2,9676	2,9303	2,9038	2,8838	2,8681	2,8554	2,8449
30	3,3546	3,1031	3,0049	2,9501	2,9144	2,8890	2,8698	2,8547	2,8425	2,8324
35	3,2762	3,0522	2,9638	2,9143	2,8818	2,8586	2,8411	2,8273	2,8161	2,8068
40	3,2160	3,0128	2,9320	2,8864	2,8564	2,8350	2,8188	2,8059	2,7955	2,7869
45	3,1680	2,9812	2,9063	2,8640	2,8361	2,8160	2,8008	2,7888	2,7791	2,7709
50	3,1288	2,9552	2,8852	2,8455	2,8193	2,8004	2,7861	2,7748	2,7655	2,7578
60	3,0681	2,9147	2,8523	2,8166	2,7931	2,7761	2,7631	2,7528	2,7445	2,7375
70	3,0228	2,8843	2,8275	2,7950	2,7734	2,7578	2,7459	2,7364	2,7287	2,7223
80	2,9876	2,8605	2,8081	2,7780	2,7580	2,7435	2,7324	2,7236	2,7164	2,7104
90	2,9591	2,8413	2,7924	2,7643	2,7456	2,7320	2,7216	2,7133	2,7065	2,7009
100	2,9356	2,8253	2,7794	2,7529	2,7352	2,7224	2,7126	2,7048	2,6984	2,6930
150	2,8593	2,7732	2,7369	2,7158	2,7016	2,6913	2,6834	2,6771	2,6719	2,6676
200	2,8163	2,7436	2,7127	2,6947	2,6826	2,6738	2,6670	2,6616	2,6571	2,6533
250	2,7879	2,7240	2,6968	2,6808	2,6701	2,6622	2,6562	2,6513	2,6473	2,6440
300	2,7675	2,7099	2,6852	2,6708	2,6610	2,6539	2,6484	2,6440	2,6404	2,6373
400	2,7395	2,6905	2,6694	2,6570	2,6486	2,6425	2,6377	2,6339	2,6308	2,6282
500	2,7208	2,6775	2,6588	2,6478	2,6403	2,6349	2,6307	2,6273	2,6245	2,6221
1 000	2,6760	2,6462	2,6333	2,6256	2,6205	2,6166	2,6137	2,6113	2,6094	2,6077
2 000	2,6455	2,6249	2,6159	2,6105	2,6069	2,6042	2,6022	2,6005	2,5991	2,5980
5 000	2,6193	2,6065	2,6009	2,5975	2,5952	2,5936	2,5923	2,5912	2,5904	2,5896
10 000	2,6064	2,5974	2,5934	2,5911	2,5895	2,5883	2,5874	2,5867	2,5860	2,5855
20 000	2,5973	2,5910	2,5882	2,5866	2,5855	2,5846	2,5840	2,5835	2,5830	2,5827
∞	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759

Table D.7 — Confidence level 99,0 % and proportion 90,0 % $(1 - \alpha = 0{,}99;\ p = 0{,}90)$

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	155,5690	19,7425	10,2697	7,4789	6,2048	5,4874	5,0311	4,7170	4,4884	4,3150
3	18,7825	7,0392	5,1183	4,3676	3,9720	3,7293	3,5660	3,4492	3,3617	3,2939
4	9,4162	4,9212	3,9582	3,5449	3,3166	3,1727	3,0742	3,0028	2,9489	2,9068
5	6,6550	4,0660	3,4311	3,1453	2,9835	2,8800	2,8086	2,7565	2,7170	2,6860

Table D.7 *(continued)*

n	m									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
6	5,3832	3,5984	3,1231	2,9026	2,7757	2,6938	2,6369	2,5953	2,5636	2,5388
7	4,6576	3,3006	2,9183	2,7369	2,6314	2,5628	2,5149	2,4798	2,4530	2,4319
8	4,1887	3,0928	2,7709	2,6156	2,5244	2,4647	2,4229	2,3922	2,3687	2,3502
9	3,8602	2,9387	2,6590	2,5223	2,4414	2,3882	2,3507	2,3231	2,3020	2,2853
10	3,6167	2,8193	2,5709	2,4481	2,3748	2,3265	2,2923	2,2671	2,2477	2,2324
11	3,4286	2,7239	2,4994	2,3874	2,3202	2,2756	2,2440	2,2206	2,2026	2,1884
12	3,2786	2,6456	2,4402	2,3368	2,2744	2,2329	2,2033	2,1814	2,1645	2,1512
13	3,1561	2,5801	2,3902	2,2939	2,2355	2,1964	2,1686	2,1479	2,1319	2,1192
14	3,0538	2,5244	2,3474	2,2569	2,2019	2,1649	2,1385	2,1188	2,1036	2,0915
15	2,9672	2,4763	2,3102	2,2248	2,1726	2,1374	2,1122	2,0934	2,0788	2,0672
16	2,8926	2,4344	2,2776	2,1965	2,1468	2,1132	2,0890	2,0709	2,0569	2,0458
17	2,8278	2,3975	2,2488	2,1715	2,1239	2,0917	2,0684	2,0510	2,0374	2,0267
18	2,7708	2,3647	2,2231	2,1491	2,1034	2,0724	2,0500	2,0331	2,0200	2,0095
19	2,7203	2,3354	2,2000	2,1290	2,0850	2,0550	2,0334	2,0170	2,0043	1,9941
20	2,6752	2,3089	2,1791	2,1108	2,0683	2,0393	2,0183	2,0024	1,9900	1,9801
22	2,5979	2,2631	2,1429	2,0791	2,0393	2,0120	1,9921	1,9770	1,9652	1,9558
24	2,5340	2,2247	2,1124	2,0525	2,0148	1,9889	1,9700	1,9556	1,9443	1,9352
26	2,4801	2,1920	2,0864	2,0297	1,9939	1,9692	1,9511	1,9373	1,9264	1,9177
28	2,4340	2,1638	2,0638	2,0099	1,9758	1,9521	1,9348	1,9215	1,9110	1,9025
30	2,3940	2,1391	2,0441	1,9926	1,9599	1,9372	1,9205	1,9076	1,8975	1,8893
35	2,3137	2,0891	2,0040	1,9575	1,9277	1,9069	1,8915	1,8796	1,8702	1,8625
40	2,2529	2,0507	1,9732	1,9304	1,9030	1,8837	1,8693	1,8582	1,8493	1,8421
45	2,2050	2,0202	1,9486	1,9089	1,8833	1,8652	1,8517	1,8412	1,8328	1,8259
50	2,1660	1,9953	1,9285	1,8913	1,8672	1,8502	1,8374	1,8274	1,8194	1,8128
60	2,1063	1,9567	1,8974	1,8641	1,8424	1,8269	1,8153	1,8062	1,7989	1,7928
70	2,0623	1,9280	1,8742	1,8439	1,8240	1,8098	1,7990	1,7906	1,7838	1,7781
80	2,0282	1,9056	1,8562	1,8281	1,8097	1,7964	1,7864	1,7785	1,7721	1,7668
90	2,0009	1,8876	1,8416	1,8154	1,7982	1,7858	1,7763	1,7689	1,7629	1,7578
100	1,9784	1,8727	1,8296	1,8050	1,7887	1,7770	1,7680	1,7610	1,7552	1,7505
150	1,9061	1,8245	1,7906	1,7711	1,7581	1,7486	1,7414	1,7357	1,7310	1,7270
200	1,8657	1,7973	1,7686	1,7520	1,7409	1,7328	1,7266	1,7216	1,7176	1,7142
250	1,8392	1,7794	1,7541	1,7394	1,7296	1,7224	1,7168	1,7124	1,7088	1,7058
300	1,8202	1,7665	1,7437	1,7304	1,7214	1,7149	1,7099	1,7059	1,7026	1,6998
400	1,7943	1,7488	1,7293	1,7179	1,7103	1,7047	1,7003	1,6969	1,6940	1,6916
500	1,7771	1,7369	1,7197	1,7097	1,7029	1,6979	1,6940	1,6909	1,6884	1,6862
1 000	1,7359	1,7086	1,6967	1,6897	1,6850	1,6815	1,6788	1,6767	1,6749	1,6734
2 000	1,7081	1,6892	1,6810	1,6762	1,6729	1,6704	1,6685	1,6670	1,6658	1,6647
5 000	1,6842	1,6726	1,6675	1,6644	1,6624	1,6608	1,6597	1,6587	1,6579	1,6573
10 000	1,6725	1,6643	1,6608	1,6586	1,6572	1,6561	1,6553	1,6546	1,6541	1,6536
20 000	1,6643	1,6586	1,6561	1,6546	1,6535	1,6528	1,6522	1,6517	1,6513	1,6510
∞	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449

Table D.8 — Confidence level 99,0 % and proportion 95,0 % ($1 - \alpha = 0,99$; $p = 0,95$)

n	m									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	182,7201	23,1159	11,9855	8,7010	7,1975	6,3481	5,8059	5,4311	5,1573	4,9489
3	22,1308	8,2618	5,9854	5,0908	4,6163	4,3233	4,1249	3,9820	3,8745	3,7907
4	11,1178	5,7889	4,6406	4,1439	3,8673	3,6914	3,5701	3,4816	3,4143	3,3616
5	7,8698	4,7921	4,0321	3,6869	3,4897	3,3624	3,2737	3,2086	3,1589	3,1198
6	6,3735	4,2479	3,6775	3,4103	3,2552	3,1542	3,0833	3,0311	2,9911	2,9596
7	5,5196	3,9016	3,4420	3,2221	3,0929	3,0081	2,9484	2,9043	2,8704	2,8436
8	4,9677	3,6599	3,2727	3,0843	2,9726	2,8989	2,8468	2,8082	2,7784	2,7550
9	4,5810	3,4807	3,1443	2,9784	2,8793	2,8136	2,7670	2,7324	2,7057	2,6846
10	4,2942	3,3419	3,0430	2,8940	2,8045	2,7449	2,7024	2,6708	2,6464	2,6271
11	4,0727	3,2308	2,9608	2,8251	2,7430	2,6881	2,6489	2,6196	2,5970	2,5791
12	3,8959	3,1396	2,8927	2,7674	2,6913	2,6403	2,6037	2,5764	2,5552	2,5384
13	3,7514	3,0633	2,8350	2,7185	2,6473	2,5994	2,5650	2,5393	2,5193	2,5034
14	3,6309	2,9983	2,7856	2,6763	2,6093	2,5640	2,5315	2,5070	2,4881	2,4730
15	3,5286	2,9422	2,7427	2,6395	2,5761	2,5331	2,5021	2,4788	2,4606	2,4462
16	3,4406	2,8932	2,7050	2,6072	2,5468	2,5057	2,4761	2,4537	2,4364	2,4225
17	3,3641	2,8501	2,6716	2,5784	2,5207	2,4814	2,4529	2,4314	2,4147	2,4013
18	3,2968	2,8117	2,6418	2,5527	2,4973	2,4596	2,4321	2,4114	2,3952	2,3822
19	3,2372	2,7774	2,6150	2,5295	2,4763	2,4399	2,4134	2,3933	2,3776	2,3650
20	3,1838	2,7464	2,5908	2,5086	2,4572	2,4220	2,3963	2,3769	2,3616	2,3494
22	3,0924	2,6926	2,5486	2,4720	2,4239	2,3908	2,3666	2,3482	2,3337	2,3221
24	3,0168	2,6475	2,5131	2,4411	2,3957	2,3644	2,3414	2,3239	2,3101	2,2989
26	2,9530	2,6091	2,4826	2,4146	2,3716	2,3417	2,3198	2,3030	2,2898	2,2791
28	2,8984	2,5759	2,4563	2,3916	2,3506	2,3221	2,3011	2,2850	2,2722	2,2619
30	2,8510	2,5468	2,4332	2,3715	2,3322	2,3049	2,2846	2,2691	2,2568	2,2468
35	2,7558	2,4878	2,3861	2,3304	2,2947	2,2697	2,2511	2,2368	2,2254	2,2161
40	2,6836	2,4425	2,3498	2,2987	2,2658	2,2427	2,2254	2,2120	2,2013	2,1926
45	2,6267	2,4064	2,3209	2,2735	2,2428	2,2211	2,2049	2,1923	2,1822	2,1739
50	2,5805	2,3768	2,2971	2,2527	2,2239	2,2035	2,1881	2,1762	2,1666	2,1587
60	2,5095	2,3311	2,2603	2,2206	2,1947	2,1762	2,1623	2,1514	2,1426	2,1353
70	2,4571	2,2970	2,2329	2,1967	2,1729	2,1559	2,1431	2,1330	2,1249	2,1181
80	2,4165	2,2705	2,2115	2,1780	2,1560	2,1402	2,1282	2,1188	2,1112	2,1048
90	2,3840	2,2491	2,1942	2,1630	2,1424	2,1276	2,1163	2,1074	2,1002	2,0942
100	2,3573	2,2314	2,1799	2,1506	2,1311	2,1171	2,1065	2,0981	2,0912	2,0855
150	2,2712	2,1740	2,1336	2,1103	2,0948	2,0835	2,0749	2,0681	2,0625	2,0578
200	2,2231	2,1416	2,1074	2,0876	2,0743	2,0647	2,0573	2,0514	2,0465	2,0425
250	2,1915	2,1203	2,0901	2,0726	2,0609	2,0523	2,0457	2,0405	2,0361	2,0325
300	2,1689	2,1049	2,0777	2,0618	2,0512	2,0434	2,0374	2,0326	2,0287	2,0254
400	2,1380	2,0838	2,0606	2,0470	2,0379	2,0312	2,0261	2,0219	2,0185	2,0157
500	2,1175	2,0697	2,0492	2,0372	2,0291	2,0231	2,0185	2,0149	2,0118	2,0093
1 000	2,0684	2,0359	2,0218	2,0134	2,0078	2,0037	2,0005	1,9979	1,9958	1,9940
2 000	2,0353	2,0128	2,0030	1,9973	1,9933	1,9904	1,9882	1,9864	1,9849	1,9836
5 000	2,0069	1,9930	1,9869	1,9833	1,9808	1,9790	1,9776	1,9765	1,9755	1,9747
10 000	1,9929	1,9832	1,9789	1,9764	1,9746	1,9734	1,9724	1,9716	1,9709	1,9704

Table D.8 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
20 000	1,9831	1,9763	1,9733	1,9715	1,9703	1,9694	1,9687	1,9682	1,9677	1,9673
∞	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600

Table D.9 — Confidence level 99,0 % and proportion 99,0 % (1 − α = 0,99; p = 0,99)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	234,8775	29,6006	15,2876	11,0563	9,1134	8,0113	7,3045	6,8136	6,4531	6,1774
3	28,5857	10,6204	7,6599	6,4888	5,8628	5,4728	5,2065	5,0131	4,8663	4,7512
4	14,4054	7,4658	5,9599	5,3025	4,9324	4,6945	4,5286	4,4063	4,3126	4,2384
5	10,2201	6,1969	5,1946	4,7343	4,4681	4,2942	4,1716	4,0806	4,0105	3,9547
6	8,2916	5,5053	4,7503	4,3924	4,1820	4,0431	3,9445	3,8709	3,8140	3,7687
7	7,1908	5,0656	4,4559	4,1605	3,9847	3,8678	3,7844	3,7220	3,6736	3,6350
8	6,4791	4,7591	4,2445	3,9911	3,8389	3,7371	3,6643	3,6096	3,5670	3,5331
9	5,9802	4,5318	4,0843	3,8610	3,7260	3,6352	3,5700	3,5210	3,4828	3,4523
10	5,6102	4,3557	3,9580	3,7574	3,6354	3,5531	3,4938	3,4491	3,4142	3,3863
11	5,3242	4,2147	3,8554	3,6727	3,5609	3,4852	3,4305	3,3893	3,3570	3,3312
12	5,0960	4,0989	3,7702	3,6018	3,4983	3,4280	3,3771	3,3386	3,3085	3,2844
13	4,9093	4,0019	3,6982	3,5415	3,4448	3,3790	3,3312	3,2951	3,2667	3,2440
14	4,7535	3,9192	3,6363	3,4895	3,3986	3,3365	3,2914	3,2572	3,2303	3,2088
15	4,6212	3,8478	3,5825	3,4441	3,3581	3,2992	3,2564	3,2238	3,1983	3,1777
16	4,5074	3,7855	3,5352	3,4040	3,3223	3,2662	3,2254	3,1942	3,1698	3,1501
17	4,4084	3,7304	3,4933	3,3684	3,2904	3,2368	3,1976	3,1678	3,1443	3,1254
18	4,3212	3,6815	3,4558	3,3365	3,2618	3,2103	3,1727	3,1440	3,1213	3,1031
19	4,2439	3,6376	3,4220	3,3077	3,2359	3,1864	3,1501	3,1224	3,1005	3,0829
20	4,1748	3,5979	3,3915	3,2816	3,2124	3,1646	3,1296	3,1027	3,0816	3,0644
22	4,0563	3,5291	3,3381	3,2359	3,1713	3,1265	3,0935	3,0682	3,0483	3,0321
24	3,9581	3,4713	3,2931	3,1972	3,1364	3,0941	3,0629	3,0389	3,0199	3,0045
26	3,8752	3,4220	3,2545	3,1639	3,1063	3,0662	3,0365	3,0136	2,9955	2,9807
28	3,8042	3,3792	3,2209	3,1350	3,0801	3,0418	3,0135	2,9916	2,9742	2,9600
30	3,7425	3,3418	3,1915	3,1095	3,0571	3,0204	2,9932	2,9721	2,9554	2,9417
35	3,6185	3,2656	3,1312	3,0574	3,0099	2,9765	2,9516	2,9323	2,9169	2,9043
40	3,5244	3,2070	3,0847	3,0171	2,9733	2,9425	2,9194	2,9015	2,8871	2,8753
45	3,4502	3,1602	3,0474	2,9847	2,9440	2,9152	2,8936	2,8768	2,8632	2,8521
50	3,3898	3,1218	3,0167	2,9581	2,9199	2,8928	2,8724	2,8565	2,8437	2,8331
60	3,2970	3,0623	2,9691	2,9167	2,8824	2,8580	2,8395	2,8250	2,8133	2,8037
70	3,2284	3,0179	2,9334	2,8857	2,8544	2,8319	2,8150	2,8016	2,7908	2,7818
80	3,1753	2,9832	2,9056	2,8615	2,8325	2,8116	2,7958	2,7834	2,7732	2,7648
90	3,1327	2,9552	2,8831	2,8420	2,8148	2,7953	2,7804	2,7687	2,7592	2,7512
100	3,0976	2,9321	2,8644	2,8258	2,8002	2,7817	2,7677	2,7566	2,7475	2,7400
150	2,9847	2,8569	2,8038	2,7732	2,7527	2,7379	2,7266	2,7176	2,7102	2,7041
200	2,9215	2,8144	2,7695	2,7434	2,7260	2,7133	2,7036	2,6958	2,6894	2,6841
250	2,8801	2,7864	2,7468	2,7238	2,7084	2,6971	2,6884	2,6815	2,6758	2,6711
300	2,8504	2,7662	2,7305	2,7096	2,6956	2,6854	2,6775	2,6713	2,6661	2,6617
400	2,8098	2,7385	2,7080	2,6902	2,6782	2,6694	2,6627	2,6572	2,6528	2,6490

Table D.9 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
500	2,7828	2,7200	2,6931	2,6773	2,6666	2,6588	2,6528	2,6479	2,6440	2,6406
1 000	2,7184	2,6756	2,6570	2,6461	2,6387	2,6332	2,6290	2,6257	2,6229	2,6205
2 000	2,6748	2,6453	2,6324	2,6248	2,6197	2,6158	2,6129	2,6105	2,6086	2,6069
5 000	2,6374	2,6192	2,6112	2,6065	2,6032	2,6008	2,5990	2,5975	2,5963	2,5952
10 000	2,6191	2,6063	2,6007	2,5974	2,5951	2,5934	2,5921	2,5911	2,5902	2,5895
20 000	2,6062	2,5973	2,5934	2,5910	2,5894	2,5882	2,5873	2,5866	2,5860	2,5855
∞	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759

Table D.10 — Confidence level 99,9 % and proportion 90,0 % ($1 - \alpha = 0{,}999$; $p = 0{,}90$)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	1555,7340	62,5942	22,3691	13,5933	10,1615	8,4070	7,3630	6,6785	6,1986	5,8452
3	59,5426	12,7713	7,8069	6,1415	5,3341	4,8647	4,5605	4,3485	4,1926	4,0734
4	20,4870	7,4872	5,3963	4,5921	4,1750	3,9224	3,7543	3,6346	3,5453	3,4760
5	12,0557	5,6774	4,4228	3,9067	3,6300	3,4592	3,3439	3,2610	3,1986	3,1500
6	8,7591	4,7730	3,8891	3,5106	3,3035	3,1742	3,0863	3,0227	2,9746	2,9369
7	7,0628	4,2289	3,5480	3,2483	3,0821	2,9775	2,9060	2,8541	2,8148	2,7839
8	6,0427	3,8639	3,3091	3,0597	2,9202	2,8318	2,7712	2,7271	2,6936	2,6672
9	5,3650	3,6009	3,1312	2,9167	2,7957	2,7187	2,6658	2,6272	2,5978	2,5747
10	4,8829	3,4016	2,9930	2,8039	2,6964	2,6279	2,5806	2,5461	2,5199	2,4992
11	4,5224	3,2450	2,8821	2,7122	2,6152	2,5531	2,5101	2,4788	2,4549	2,4361
12	4,2426	3,1183	2,7909	2,6362	2,5473	2,4902	2,4507	2,4219	2,3999	2,3826
13	4,0189	3,0135	2,7145	2,5719	2,4896	2,4366	2,3999	2,3730	2,3525	2,3364
14	3,8358	2,9253	2,6494	2,5167	2,4398	2,3902	2,3558	2,3306	2,3113	2,2962
15	3,6830	2,8499	2,5932	2,4689	2,3965	2,3497	2,3171	2,2933	2,2751	2,2608
16	3,5536	2,7845	2,5441	2,4269	2,3583	2,3139	2,2830	2,2603	2,2430	2,2294
17	3,4423	2,7274	2,5009	2,3897	2,3245	2,2821	2,2525	2,2309	2,2143	2,2013
18	3,3456	2,6769	2,4624	2,3566	2,2942	2,2536	2,2252	2,2044	2,1885	2,1760
19	3,2607	2,6319	2,4280	2,3268	2,2670	2,2279	2,2006	2,1805	2,1652	2,1532
20	3,1856	2,5916	2,3970	2,3000	2,2424	2,2046	2,1783	2,1589	2,1441	2,1324
22	3,0583	2,5221	2,3434	2,2533	2,1995	2,1641	2,1393	2,1210	2,1070	2,0960
24	2,9544	2,4644	2,2984	2,2141	2,1634	2,1299	2,1064	2,0890	2,0757	2,0652
26	2,8678	2,4155	2,2602	2,1807	2,1326	2,1007	2,0782	2,0616	2,0489	2,0388
28	2,7944	2,3736	2,2273	2,1519	2,1060	2,0755	2,0539	2,0379	2,0256	2,0159
30	2,7313	2,3371	2,1986	2,1267	2,0828	2,0534	2,0326	2,0171	2,0052	1,9958
35	2,6061	2,2636	2,1405	2,0757	2,0358	2,0088	1,9894	1,9750	1,9639	1,9551
40	2,5127	2,2077	2,0962	2,0368	1,9999	1,9747	1,9566	1,9430	1,9324	1,9241
45	2,4399	2,1636	2,0611	2,0061	1,9715	1,9478	1,9307	1,9177	1,9077	1,8996
50	2,3814	2,1278	2,0326	1,9810	1,9485	1,9260	1,9097	1,8973	1,8876	1,8799
60	2,2925	2,0727	1,9886	1,9426	1,9132	1,8927	1,8777	1,8662	1,8571	1,8499
70	2,2276	2,0321	1,9562	1,9142	1,8873	1,8683	1,8543	1,8435	1,8350	1,8281
80	2,1779	2,0006	1,9310	1,8923	1,8673	1,8496	1,8364	1,8262	1,8181	1,8115
90	2,1383	1,9754	1,9109	1,8748	1,8513	1,8347	1,8222	1,8125	1,8048	1,7985
100	2,1059	1,9546	1,8943	1,8603	1,8382	1,8224	1,8106	1,8014	1,7940	1,7879

Table D.10 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
150	2,0029	1,8878	1,8408	1,8140	1,7963	1,7835	1,7739	1,7662	1,7601	1,7549
200	1,9461	1,8504	1,8109	1,7881	1,7730	1,7621	1,7537	1,7471	1,7417	1,7372
250	1,9091	1,8259	1,7912	1,7711	1,7578	1,7481	1,7406	1,7347	1,7299	1,7258
300	1,8827	1,8083	1,7771	1,7590	1,7468	1,7380	1,7313	1,7259	1,7215	1,7178
400	1,8469	1,7842	1,7577	1,7423	1,7319	1,7244	1,7185	1,7139	1,7101	1,7069
500	1,8232	1,7682	1,7449	1,7312	1,7220	1,7153	1,7101	1,7060	1,7026	1,6997
1 000	1,7671	1,7300	1,7140	1,7046	1,6982	1,6936	1,6900	1,6871	1,6847	1,6827
2 000	1,7294	1,7040	1,6930	1,6865	1,6820	1,6788	1,6763	1,6743	1,6726	1,6712
5 000	1,6974	1,6817	1,6749	1,6709	1,6681	1,6661	1,6645	1,6632	1,6622	1,6613
10 000	1,6817	1,6708	1,6660	1,6631	1,6612	1,6598	1,6587	1,6578	1,6571	1,6564
20 000	1,6707	1,6631	1,6597	1,6577	1,6564	1,6554	1,6546	1,6540	1,6535	1,6530
∞	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449	1,6449

Table D.11 — Confidence level 99,9 % and proportion 95,0 % ($1 - \alpha = 0,999$; $p = 0,95$)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	1827,2522	73,2838	26,0939	15,7955	11,7620	9,6947	8,4608	7,6494	7,0787	6,6574
3	70,1538	14,9785	9,1103	7,1319	6,1666	5,6019	5,2338	4,9760	4,7860	4,6403
4	24,1850	8,7950	6,3062	5,3407	4,8352	4,5266	4,3198	4,1720	4,0613	3,9754
5	14,2518	6,6792	5,1776	4,5531	4,2145	4,0035	3,8602	3,7567	3,6785	3,6175
6	10,3659	5,6230	4,5609	4,1002	3,8451	3,6842	3,5740	3,4939	3,4332	3,3856
7	8,3658	4,9882	4,1678	3,8015	3,5958	3,4650	3,3748	3,3091	3,2592	3,2199
8	7,1627	4,5627	3,8928	3,5874	3,4141	3,3032	3,2265	3,1704	3,1277	3,0940
9	6,3633	4,2562	3,6884	3,4253	3,2747	3,1779	3,1107	3,0615	3,0240	2,9944
10	5,7945	4,0241	3,5298	3,2976	3,1638	3,0774	3,0174	2,9733	2,9397	2,9131
11	5,3691	3,8417	3,4025	3,1939	3,0730	2,9947	2,9402	2,9001	2,8695	2,8453
12	5,0388	3,6941	3,2979	3,1079	2,9972	2,9252	2,8751	2,8382	2,8099	2,7877
13	4,7747	3,5721	3,2102	3,0351	2,9327	2,8659	2,8193	2,7850	2,7587	2,7380
14	4,5585	3,4692	3,1354	2,9727	2,8771	2,8146	2,7709	2,7387	2,7140	2,6946
15	4,3780	3,3813	3,0708	2,9185	2,8286	2,7697	2,7285	2,6980	2,6747	2,6563
16	4,2251	3,3050	3,0144	2,8709	2,7858	2,7300	2,6909	2,6620	2,6399	2,6224
17	4,0936	3,2383	2,9646	2,8287	2,7479	2,6947	2,6574	2,6298	2,6086	2,5919
18	3,9793	3,1793	2,9204	2,7910	2,7139	2,6630	2,6272	2,6008	2,5805	2,5645
19	3,8789	3,1268	2,8807	2,7572	2,6833	2,6344	2,6000	2,5746	2,5551	2,5396
20	3,7900	3,0796	2,8449	2,7266	2,6555	2,6085	2,5753	2,5507	2,5319	2,5170
22	3,6394	2,9983	2,7829	2,6733	2,6071	2,5632	2,5320	2,5090	2,4912	2,4772
24	3,5164	2,9307	2,7309	2,6285	2,5663	2,5248	2,4954	2,4735	2,4567	2,4434
26	3,4138	2,8734	2,6866	2,5901	2,5313	2,4919	2,4639	2,4431	2,4270	2,4143
28	3,3269	2,8241	2,6483	2,5570	2,5010	2,4634	2,4366	2,4166	2,4012	2,3890
30	3,2521	2,7812	2,6149	2,5280	2,4745	2,4384	2,4126	2,3934	2,3785	2,3667
35	3,1037	2,6947	2,5471	2,4690	2,4205	2,3876	2,3638	2,3460	2,3322	2,3212
40	2,9928	2,6288	2,4952	2,4238	2,3791	2,3486	2,3264	2,3097	2,2967	2,2863
45	2,9064	2,5767	2,4540	2,3879	2,3463	2,3176	2,2967	2,2809	2,2685	2,2586
50	2,8368	2,5343	2,4204	2,3587	2,3195	2,2924	2,2725	2,2574	2,2456	2,2361

Table D.11 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
60	2,7311	2,4691	2,3686	2,3135	2,2783	2,2536	2,2355	2,2216	2,2106	2,2017
70	2,6540	2,4209	2,3303	2,2801	2,2478	2,2251	2,2083	2,1953	2,1849	2,1766
80	2,5949	2,3835	2,3005	2,2543	2,2243	2,2031	2,1873	2,1751	2,1653	2,1573
90	2,5478	2,3535	2,2766	2,2335	2,2054	2,1855	2,1706	2,1590	2,1497	2,1421
100	2,5092	2,3288	2,2569	2,2164	2,1899	2,1711	2,1569	2,1459	2,1370	2,1297
150	2,3865	2,2493	2,1933	2,1614	2,1402	2,1250	2,1135	2,1044	2,0970	2,0909
200	2,3188	2,2048	2,1577	2,1306	2,1126	2,0995	2,0896	2,0817	2,0753	2,0699
250	2,2748	2,1757	2,1343	2,1104	2,0945	2,0829	2,0740	2,0670	2,0612	2,0564
300	2,2434	2,1547	2,1175	2,0959	2,0815	2,0710	2,0629	2,0565	2,0512	2,0468
400	2,2007	2,1260	2,0944	2,0760	2,0637	2,0547	2,0478	2,0422	2,0377	2,0338
500	2,1725	2,1070	2,0791	2,0628	2,0519	2,0439	2,0377	2,0328	2,0287	2,0253
1 000	2,1056	2,0614	2,0423	2,0311	2,0235	2,0180	2,0137	2,0102	2,0074	2,0050
2 000	2,0607	2,0305	2,0173	2,0095	2,0043	2,0004	1,9974	1,9950	1,9930	1,9913
5 000	2,0225	2,0039	1,9958	1,9909	1,9877	1,9852	1,9834	1,9819	1,9806	1,9796
10 000	2,0038	1,9908	1,9851	1,9817	1,9794	1,9777	1,9764	1,9754	1,9745	1,9737
20 000	1,9908	1,9817	1,9777	1,9753	1,9737	1,9725	1,9716	1,9708	1,9702	1,9697
∞	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600	1,9600

Table D.12 — Confidence level 99,9 % and proportion 99,0 % ($1 - \alpha = 0,999$; $p = 0,99$)

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	2348,8387	93,8333	33,2653	20,0444	14,8573	12,1910	10,5938	9,5391	8,7942	8,2420
3	90,6105	19,2385	11,6321	9,0532	7,7853	7,0373	6,5458	6,1990	5,9416	5,7433
4	31,3298	11,3247	8,0703	6,7950	6,1194	5,7024	5,4200	5,2164	5,0629	4,9431
5	18,5010	8,6194	6,6422	5,8089	5,3506	5,0612	4,8622	4,7173	4,6071	4,5206
6	13,4784	7,2704	5,8646	5,2452	4,8967	4,6737	4,5189	4,4055	4,3188	4,2505
7	10,8920	6,4607	5,3703	4,8753	4,5924	4,4096	4,2820	4,1880	4,1161	4,0592
8	9,3356	5,9183	5,0256	4,6112	4,3716	4,2158	4,1065	4,0258	3,9639	3,9148
9	8,3012	5,5280	4,7697	4,4117	4,2029	4,0663	3,9702	3,8991	3,8444	3,8010
10	7,5649	5,2325	4,5713	4,2549	4,0689	3,9468	3,8606	3,7968	3,7475	3,7085
11	7,0142	5,0002	4,4124	4,1278	3,9595	3,8486	3,7702	3,7120	3,6670	3,6314
12	6,5864	4,8124	4,2817	4,0223	3,8682	3,7663	3,6940	3,6403	3,5989	3,5659
13	6,2443	4,6570	4,1722	3,9332	3,7906	3,6960	3,6288	3,5788	3,5402	3,5095
14	5,9641	4,5260	4,0788	3,8568	3,7237	3,6352	3,5723	3,5254	3,4891	3,4603
15	5,7303	4,4139	3,9981	3,7903	3,6653	3,5820	3,5226	3,4784	3,4441	3,4169
16	5,5319	4,3168	3,9276	3,7319	3,6138	3,5349	3,4787	3,4366	3,4041	3,3782
17	5,3613	4,2317	3,8654	3,6802	3,5680	3,4930	3,4394	3,3993	3,3683	3,3436
18	5,2130	4,1565	3,8099	3,6339	3,5270	3,4553	3,4041	3,3657	3,3360	3,3123
19	5,0827	4,0894	3,7602	3,5923	3,4900	3,4213	3,3721	3,3353	3,3067	3,2840
20	4,9673	4,0291	3,7154	3,5546	3,4564	3,3904	3,3430	3,3075	3,2800	3,2581
22	4,7717	3,9252	3,6375	3,4889	3,3978	3,3362	3,2920	3,2588	3,2330	3,2125
24	4,6118	3,8385	3,5720	3,4335	3,3481	3,2903	3,2486	3,2173	3,1929	3,1735
26	4,4784	3,7650	3,5161	3,3859	3,3054	3,2507	3,2112	3,1815	3,1583	3,1398
28	4,3653	3,7018	3,4677	3,3447	3,2683	3,2163	3,1786	3,1502	3,1281	3,1104

Table D.12 *(continued)*

n	*m*									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
30	4,2679	3,6466	3,4254	3,3085	3,2357	3,1860	3,1499	3,1227	3,1014	3,0844
35	4,0745	3,5352	3,3393	3,2347	3,1690	3,1239	3,0911	3,0661	3,0466	3,0310
40	3,9299	3,4501	3,2731	3,1778	3,1175	3,0759	3,0455	3,0223	3,0042	2,9895
45	3,8170	3,3827	3,2203	3,1323	3,0764	3,0376	3,0091	2,9873	2,9702	2,9563
50	3,7261	3,3277	3,1772	3,0951	3,0427	3,0061	2,9792	2,9586	2,9423	2,9291
60	3,5879	3,2430	3,1104	3,0374	2,9904	2,9574	2,9330	2,9141	2,8992	2,8870
70	3,4870	3,1802	3,0607	2,9944	2,9515	2,9213	2,8987	2,8812	2,8673	2,8559
80	3,4095	3,1314	3,0221	2,9610	2,9213	2,8932	2,8721	2,8557	2,8426	2,8319
90	3,3478	3,0923	2,9910	2,9341	2,8970	2,8706	2,8508	2,8353	2,8229	2,8127
100	3,2972	3,0600	2,9653	2,9119	2,8769	2,8520	2,8333	2,8186	2,8067	2,7970
150	3,1362	2,9559	2,8822	2,8402	2,8123	2,7923	2,7771	2,7651	2,7553	2,7472
200	3,0474	2,8975	2,8356	2,7999	2,7762	2,7590	2,7459	2,7355	2,7270	2,7200
250	2,9896	2,8592	2,8049	2,7734	2,7525	2,7372	2,7256	2,7163	2,7087	2,7024
300	2,9483	2,8317	2,7828	2,7544	2,7354	2,7216	2,7110	2,7026	2,6956	2,6898
400	2,8922	2,7940	2,7525	2,7283	2,7121	2,7003	2,6911	2,6839	2,6779	2,6729
500	2,8551	2,7690	2,7324	2,7110	2,6966	2,6861	2,6780	2,6715	2,6661	2,6616
1 000	2,7672	2,7091	2,6840	2,6693	2,6594	2,6521	2,6464	2,6419	2,6382	2,6350
2 000	2,7083	2,6685	2,6512	2,6410	2,6340	2,6290	2,6250	2,6219	2,6192	2,6170
5 000	2,6580	2,6336	2,6229	2,6165	2,6122	2,6090	2,6066	2,6046	2,6030	2,6016
10 000	2,6334	2,6164	2,6089	2,6044	2,6014	2,5992	2,5975	2,5961	2,5949	2,5939
20 000	2,6163	2,6044	2,5991	2,5960	2,5939	2,5923	2,5911	2,5901	2,5893	2,5886
∞	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759	2,5759

Annex E
(normative)

Distribution-free statistical tolerance intervals

See Tables E.1 and E.2.

Table E.1 — Distribution-free statistical tolerance intervals — Sample size *n* for a proportion *p* at confidence level 1 − *α*, given *v* and *w*

	Confidence level 90 % (1 − α = 0,90)				Confidence level 95 % (1 − α = 0,95)		
	Proportion p × 100 %				Proportion p × 100 %		
$v + w$	90	95	99		90	95	99
1	22	45	230		29	59	299
2	38	77	388		46	93	473
3	52	105	531		61	124	628
4	65	132	667		76	153	773
5	78	158	798		89	181	913
6	91	184	926		103	208	1049
7	104	209	1051		116	234	1182
8	116	234	1175		129	260	1312
9	128	258	1297		142	286	1441
10	140	282	1418		154	311	1568
11	152	306	1538		167	336	1693
12	164	330	1658		179	361	1818
13	175	353	1776		191	386	1941
14	187	377	1893		203	410	2064
15	199	400	2010		215	434	2185
16	210	423	2127		227	458	2306
17	222	446	2242		239	482	2426
18	233	469	2358		251	506	2546
19	245	492	2473		263	530	2665
20	256	515	2587		275	554	2784

Table E.2 — Distribution-free statistical tolerance intervals *(continued)* — Sample size *n* for a proportion *p* at confidence level 1−α, given *v* and *w*

	Confidence level 99 % (1 − α = 0,99)				Confidence level 99,9 % (1 − α = 0,999)		
	Proportion p × 100 %				Proportion p × 100 %		
$v + w$	90	95	99		90	95	99
1	44	90	459		66	135	688
2	64	130	662		89	181	920
3	81	165	838		108	220	1119
4	97	198	1001		126	257	1302
5	113	229	1157		143	291	1475
6	127	259	1307		159	324	1640
7	142	288	1453		175	356	1801
8	156	316	1596		190	387	1957
9	170	344	1736		205	417	2110
10	183	371	1874		220	447	2259
11	197	398	2010		235	476	2407
12	210	425	2144		249	505	2552
13	223	451	2277		263	533	2696
14	236	478	2409		277	562	2837
15	249	504	2539		291	590	2978
16	262	529	2669		305	617	3117
17	275	555	2798		318	645	3255
18	287	580	2925		332	672	3391
19	300	606	3052		345	699	3527
20	312	631	3179		358	726	3662

Annex F
(informative)

Computation of factors for two-sided parametric statistical tolerance intervals

In the field of mathematical statistics, the interval for the case of unknown mean μ and unknown standard deviation σ is called a p-content tolerance interval with confidence level $1 - \alpha$ for a normal distribution. The symbol β is sometimes used instead of the symbol p. Although the definition of a p-content tolerance interval is simple, the computation of precise values of tolerance factors is fairly difficult, particularly without the use of a computer. We consider the tolerance interval constructed by $[\bar{x} - k \times s, \bar{x} + k \times s]$, where $\bar{x}$ and s are, respectively, the sample mean and the sample standard deviation.

The value of a tolerance factor is the solution in k of the following integral equation

$$\sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} F(x,k)\, e^{-\frac{nx^2}{2}}\, dx - 1 + \alpha = 0, \qquad \text{(F.1)}$$

where

$$F(x,k) = \int_{\frac{f\,R^2(x)}{k^2}}^{\infty} \frac{t^{\frac{f}{2}-1} e^{-\frac{t}{2}}}{2^{\frac{f}{2}} \Gamma\left(\frac{f}{2}\right)} dt,$$

and $R(x)$ is the solution of the equation $\Phi(x+R) - \Phi(x-R) - p = 0$.

In the formula for $F(x,k)$, Formula (F.1), the symbol f is the number of degrees of freedom, which depends on the number of samples and the number of observations in each sample.

NOTE 1 For one sample of size n the degrees of freedom is $f = n - 1$.

NOTE 2 For m samples of the same size n (balanced model) the degrees of freedom is $f = m(n - 1)$.

NOTE 3 For m samples of sizes $n_1, n_2, \ldots, n_m$ (unbalanced model) the degrees of freedom is

$$f = \sum_{i=1}^{m} (n_i - 1) = \left(\sum_{i=1}^{m} n_i \right) - m.$$

In this case Formula (F.1) is modified; n is substituted by n_i and k by k_i and we obtain separate solutions k_i for each sample.

Analytical derivation of the solution of Formula (F.1) with respect to k is difficult, if not impossible, so approximate methods for the computation of factor k have been used in the past. In the previous standard of tolerance interval (ISO 3207:1975), the factors in the table for two-sided statistical tolerance interval for the case of unknown μ and σ were obtained by such a method.

More recently, computer programs that use numerical integration for the exact computation of the factors have been developed. In Annex D the factors, which were derived by an iterative process using numerical integration, have been calculated to give at least the required confidence level.

Extensive tables of the factor k for two-sided statistical tolerance interval for the normal distribution with unknown μ and σ have been published by Garaj and Janiga[9] with an introduction to the tables given in English, French, German, and Slovak. These tables correspond to the column m=1 of the tables of Annex D in this part of ISO 16269, but the number of entries and the ranges of n, p and α are larger in the tables by Garaj and Janiga[9].

In Annex D are given tables of the factor k for two-sided statistical tolerance intervals for the normal distributions with unknown $\mu_i (i = 1,2,...,m;\ m = 2(1)10)$ and unknown common σ.

Extensive tables of the factor k for two-sided statistical tolerance interval for the normal distribution with unknown μ and common σ have also been published by Garaj and Janiga[10] with an introduction to the tables given in English, French, German, and Slovak. These tables correspond to the columns m=2(1)10 of the tables in Annex D in this part of ISO 16269, but the number of entries, the number of decimal places and the ranges of m, n, p and α are larger in the tables by Garaj and Janiga.[10]

Annex G
(informative)

Construction of a distribution-free statistical tolerance interval for any type of distribution

G.1 Infinite populations

Assume we have a sample, $x_1, x_2, \ldots, x_n$, of independent random observations on a population (continuous, discrete, or mixed) and let its order statistics be $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$.

The interval with 100(1 −α) % confidence that at least 100p % of the population lie between the vth smallest observation (i.e., order statistic $x_{(v)}$) and the w^{th} largest observation (i.e., order statistic $x_{(n-w+1)}$) of the sample is determined by solving the cumulative binomial distribution for smallest sample size n such that: .

$$\sum_{x=0}^{v+w-1} \binom{n}{x} p^{n-x} (1-p)^x \le \alpha, \tag{G.1}$$

where $v \ge 0$, $w \ge 0$, $v + w \ge 1$, $0 < p < 1$, and $0 < \alpha < 1$.

When the cumulative distribution function of the population characteristic X is not continuous, the statement above is modified such that there is *at least* 100(1−α) % confidence that at least 100p % of the population is between $x_{(v)}$ and $x_{(n-w+1)}$ *or equal to* $x_{(v)}$ *or* $x_{(n-w+1)}$.

When $v = 0$, $x_{(0)}$ corresponds to the lower bound on X (e.g. −4) and the associated interval is referred to as a one-sided upper statistical tolerance interval. When $w = 0$, $x_{(n+1)}$ corresponds to the upper bound on X (e.g. +4) and the associated interval is referred to as a one-sided lower statistical tolerance interval. When $v \ge 1$ and $w \ge 1$ the associated interval between the two order statistics is referred to as a two-sided interval. When dealing with discrete values which are either acceptable or unacceptable, set $v + w - 1$ equal to the maximum permissible number of nonconforming items in the sample.

When $v + w = 1$, Formula (G.1) reduces to:

$$p^n \le \alpha \tag{G.2}$$

When $v + w = 2$, Formula (G.1) reduces to:

$$np^{n-1} - (n-1)p^n \le \alpha \tag{G.3}$$

G.2 Finite populations

Assume we have a finite population with the values $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ associated with its N elements. A simple random sample of size n is drawn without replacement and its order statistics are $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$.

The interval with at least 100(1−α) % confidence that 100p % of the population lie between the v^{th} smallest observation (i.e., order statistic $x_{(v)}$) and the w^{th} largest observation (i.e., order statistic $x_{(n-}$

$_{w+1)}$) in the sample is determined by solving the cumulative hypergeometric distribution for smallest sample size n such that:

$$\sum_{x=0}^{v+w-1} \frac{\binom{N-M+c}{x}\binom{M-c}{n-x}}{\binom{N}{n}} \le \alpha \tag{G.4}$$

where $v \ge 0$, $w \ge 0$, $v + w \ge 1$, $0 < p < 1$, $0 < \alpha < 1$, $M = \lceil Np \rceil$ (the least integer greater than or equal to Np), and $c = 0$, 1, or 2 depending on whether the interval corresponds to a discrete value, is one-sided, or is two-sided respectively.

When $v = 0$, $x_{(0)}$ corresponds to the lower bound on X (e.g. −4) and the associated interval is referred to as a one-sided upper statistical tolerance interval. When $w = 0$, $x_{(n+1)}$ corresponds to the upper bound on X (e.g. +4) and the associated interval is referred to as a one-sided lower statistical tolerance interval. When $v \ge 1$ and $w \ge 1$ the associated interval between the two order statistics is referred to as a two-sided interval. When $c = 0$, set $v + w - 1$ equal to the maximum permissible number of nonconforming items in the sample.

Additional technical information may be found in Reference [7].

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