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Statistical interpretation of data — Techniques of estimation and tests relating to means and variances

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FOREWORD

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Statistical interpretation of data – Techniques of estimation and tests relating to means and variances

SECTION ONE : PRESENTATION OF CALCULATIONS

GENERAL REMARKS

1) This International Standard specifies the techniques required :

a) to estimate the mean or the variance of populations;

b) to examine certain hypotheses concerning the value of those parameters, from samples.

2) The techniques used are valid only if, in each of the populations under consideration, the sample elements are drawn at random and are independent. In the case of a finite population, elements drawn at random may be considered as independent when the population size is sufficiently large or when the sampling fraction is sufficiently small (for instance smaller than 1/10).

3) The distribution of the observed variable is assumed to be normal in each population. However, if the distribution does not deviate very much from the normal, the techniques described remain approximately valid to an extent sufficient for most practical applications, provided the sample size is not too small. For tables A, B, C and D, the sample size should be of the order of 5 to 10 at least; for all the other tables, it should be not less than about 20.[1)]

4) A certain number of techniques exist which permit the verification of the hypothesis of normality. This subject is dealt with briefly in the examples in section two and will also be dealt with in a further document (yet to be prepared). Nevertheless, this hypothesis may be admitted on the basis of information other than that provided by the sample itself. In the case where the hypothesis of normality should be rejected, the obvious method to follow is to resort to non-parametric tests or to use suitable transformations for obtaining normally distributed populations, for example $1/x$, $\log(x + a)$, $\sqrt{x + a}$, but the conclusions reached by applying these procedures described in this International Standard are only directly valid for the transformed variate; caution should be used in the translation to the original variate. For example **exp (mean log x)** is equal to the **geometric mean** of x not the arithmetic mean.

If what is really needed is an estimate of the mean or standard deviation of the variate X itself then, whether the population distribution is normal or not, an unbiassed estimation of the mean m and the population variance σ^2 is produced by the sample mean $\bar{x}$ and characteristic s^2.

5) It is desirable to accompany each statistical operation with all the particulars relevant to the source or to the method of obtaining the observations which may clarify this statistical analysis, and in particular to give the unit or the smallest unit of measurement having practical meaning.

6) It is not permissible to discard any observations or to apply any corrections to apparently doubtful observations without a justification based on experimental, technical or other evident grounds which should be clearly given. In any case the discarded or corrected values and the reason for discarding or correcting them must be mentioned.

7) In problems of estimation, the confidence level $1 - \alpha$ is the probability that the confidence interval covers the true value of the estimated parameter. Its most usual values are 0,95 and 0,99, or $\alpha = 0{,}05$ and $\alpha = 0{,}01$.

8) In problems of testing a hypothesis, the significance level is, in the two-sided cases, the probability of rejecting the null hypothesis (or tested hypothesis) if it is true (error of the first kind); in the one-sided cases, the significance level is the maximum value of this probability (maximum value of the error of the first kind). Values of $\alpha = 0{,}05$ (1 in 20 chance) or 0,01 (1 in 100 chance) are very commonly employed according to the risk which the user is prepared to take. Since a hypothesis may be rejected using $\alpha = 0{,}05$, but not when using 0,01, it is often appropriate to use the phrase : "the hypothesis is rejected at the 5 % level" or, if this is the case, "at the 1 % level". Attention is drawn to the existence of an error of the second kind. This error is committed if the null hypothesis is accepted when it is false. Terms concerning statistical tests are defined in clause 2 of ISO 3534, *Statistics – Vocabulary*[2)].

1) Studies about normal distributions are in progress in TC 69/SC 2.

2) At present at the stage of draft.

9) The calculations can often be greatly reduced by making a change of origin and/or unit on the data. In the case of data classified into groups, reference may be made to the formulae in ISO 2602, *Statistical interpretation of test results — Estimation of the mean — Confidence interval.*

NOTE — A change of origin may be essential to obtain sufficient accuracy when calculating a variance using the stated formulae with a low precision calculator or computer.

10) The methods shown in tables C and C′ deal with the comparison of two means. They assume that the corresponding samples are independent. For the study of certain problems, it may be interesting to pair the observations (for instance in the comparison of two methods or the comparison of two instruments). The statistical treatment of paired observations is the subject of ISO 3301, *Statistical interpretation of data — Comparison of two means in the case of paired observations*, but in annex A an example of treatment of paired observations is given. It uses formally the data of table A′.

11) The symbols and their definitions used in this International Standard are in conformity with ISO 3207, *Statistical interpretation of data — Determination of a statistical tolerance interval.*

TABLES

TABLE A — **Comparison of a mean with a given value** (variance known)

Technical characteristics of the population studied (5) . . . .

Technical characteristics of the sample items (5) . . . .

Discarded observations (6) . . . .

Statistical data	Calculations
Sample size : $n =$	$\bar{x} = \frac{\Sigma x}{n} =$
Sum of the observed values : $\Sigma x =$	$[u_{1-\alpha}/\sqrt{n}]\ \sigma =$
Given value : $m_0 =$	$[u_{1-\alpha/2}/\sqrt{n}]\ \sigma =$
Known value of the population variance : $\sigma^2 =$	
Or standard deviation : $\sigma =$	
Significance level chosen (8) : $\alpha =$	

Results

Comparison of the population mean with the given value m_0 :

Two-sided case :

The hypothesis of the equality of the population mean to the given value (null hypothesis) is rejected if :

$$|\bar{x} - m_0| > [u_{1-\alpha/2}/\sqrt{n}]\ \sigma$$

One-sided cases :

a) The hypothesis that the population mean is not smaller than m_0 (null hypothesis) is rejected if :

$$\bar{x} < m_0 - [u_{1-\alpha}/\sqrt{n}]\ \sigma$$

b) The hypothesis that the population mean is not greater than m_0 (null hypothesis) is rejected if :

$$\bar{x} > m_0 + [u_{1-\alpha}/\sqrt{n}]\ \sigma$$

NOTE — The numbers (5), (6) and (8) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The significance level α (see § 8 of the "General remarks") is the probability of rejecting the null hypothesis when this hypothesis is true.

2) U stands for the standardized normal variate : the value u_α is defined by :

$$P\,[U < u_\alpha] = \alpha$$

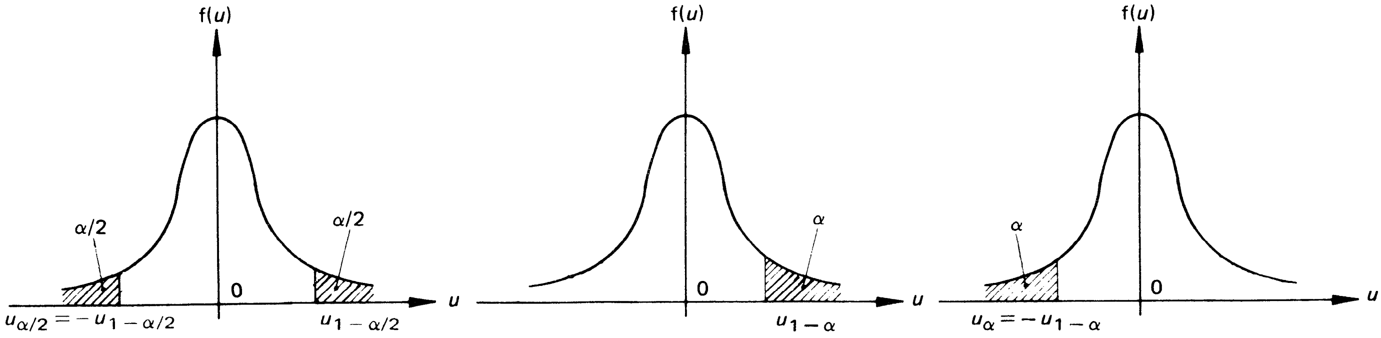
Since the distribution of U is symmetrical around zero, $u_\alpha = -u_{1-\alpha}$.

We therefore have :

$$P\,[U > u_\alpha] = 1 - \alpha$$

$$P\,[-u_{1-\alpha/2} < U < u_{1-\alpha/2}] = 1 - \alpha$$

Probability density of U (standardized normal distribution)



Two-sided case — One-sided cases

3) $\sigma/\sqrt{n}$ is the standard deviation of the mean $\bar{x}$, in a sample of n observations.

4) For convenience in application, values of $u_{1-\alpha}/\sqrt{n}$ and $u_{1-\alpha/2}/\sqrt{n}$ are given in table 1 of annex B for $\alpha = 0{,}05$ and $\alpha = 0{,}01$.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE A' – **Comparison of a mean with a given value** (variance unknown)

Technical characteristics of the population studied (5) . . . .

Technical characteristics of the sample items (5) . . . .

Discarded observations (6) . . . .

Statistical data	**Calculations**
Sample size : $n =$	$\bar{x} = \frac{\Sigma x}{n} =$
Sum of the observed values : $\Sigma x =$	$\frac{\Sigma (x - \bar{x})^2}{n-1} = \frac{\Sigma x^2 - (\Sigma x)^2/n}{n-1}$
Sum of the squares of the observed values : $\Sigma x^2 =$	$\sigma^* = s = \sqrt{\frac{\Sigma (x - \bar{x})^2}{n-1}} =$
Given value : $m_0 =$	$[t_{1-\alpha}(\nu)/\sqrt{n}\,]\, s =$
Degrees of freedom : $\nu = n - 1$	$[t_{1-\alpha/2}(\nu)/\sqrt{n}\,]\, s =$
Significance level chosen (8) : $\alpha =$	

Results

Comparison of the population mean with the given value m_0 :

Two-sided case :

The hypothesis of the equality of the population mean to the given value (null hypothesis) is rejected if :

$$|\bar{x} - m_0| > [t_{1-\alpha/2}(\nu)/\sqrt{n}\,]\, s$$

One-sided cases :

a) The hypothesis that the population mean is not smaller than m_0 (null hypothesis) is rejected if :

$$\bar{x} < m_0 - [t_{1-\alpha}(\nu)/\sqrt{n}\,]\, s$$

b) The hypothesis that the population mean is not greater than m_0 (null hypothesis) is rejected if :

$$\bar{x} > m_0 + [t_{1-\alpha}(\nu)/\sqrt{n}\,]\, s$$

NOTE – The numbers (5), (6) and (8) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The significance level α (see § 8 of the "General remarks") is the probability of rejecting the null hypothesis when this hypothesis is true.

2) $t(\nu)$ stands for Student's variate with $\nu = n - 1$ degrees of freedom : the value $t_\alpha(\nu)$ is defined by

$$P\,[t(\nu) < t_\alpha(\nu)] = \alpha$$

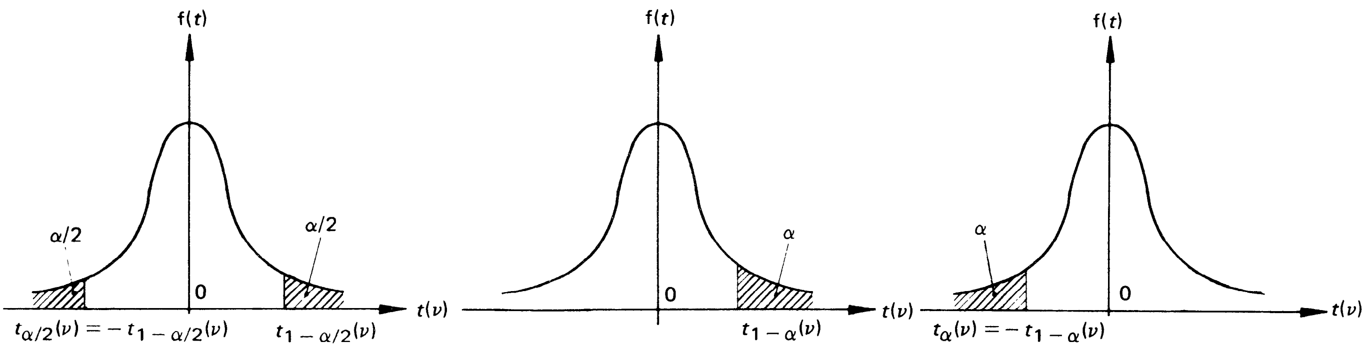
Since the distribution of $t(\nu)$ is symmetrical around zero, $t_\alpha(\nu) = -t_{1-\alpha}(\nu)$.

We therefore have :

$$P\,[t(\nu) > t_\alpha(\nu)] = 1 - \alpha$$

$$P\,[-t_{1-\alpha/2}(\nu) < t(\nu) < t_{1-\alpha/2}(\nu)] = 1 - \alpha$$

Probability density of Student's $t(\nu)$ with $\nu = n - 1$ degrees of freedom



Two-sided case

One-sided cases

3) $\sigma^*/\sqrt{n}$ is the estimated standard deviation of the mean $\bar{x}$, in a sample of n observations.

4) For convenience in application, values of $t_{1-\alpha/2}(\nu)/\sqrt{n}$ and $t_{1-\alpha}(\nu)/\sqrt{n}$ are given in table IIb of annex B for $\alpha = 0{,}05$ and $\alpha = 0{,}01$

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE B – **Estimation of a mean** (variance known)

Technical characteristics of the population studied (5) . . . .

Technical characteristics of the sample items (5) . . . .

Discarded observations (6) . . . .

Statistical data	**Calculations**
Sample size : $n =$	$\bar{x} = \frac{\Sigma x}{n} =$
Sum of the observed values : $\Sigma x =$	$[u_{1-\alpha}/\sqrt{n}]\,\sigma =$
Known value of the population variance : $\sigma^2 =$	
Or standard deviation : $\sigma =$	$[u_{1-\alpha/2}/\sqrt{n}]\,\sigma =$
Confidence level chosen (7) : $1 - \alpha =$	

Results

Estimation of the population mean m :

$$m^* = \bar{x} =$$

Two-sided confidence interval :

$$\bar{x} - [u_{1-\alpha/2}/\sqrt{n}]\,\sigma < m < \bar{x} + [u_{1-\alpha/2}/\sqrt{n}]\,\sigma$$

One-sided confidence intervals :

$$m < \bar{x} + [u_{1-\alpha}/\sqrt{n}]\,\sigma$$

or

$$m > \bar{x} - [u_{1-\alpha}/\sqrt{n}]\,\sigma$$

NOTE – The numbers (5), (6) and (7) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The confidence level $1-\alpha$ (see § 7 of the "General remarks") is the probability that the confidence interval covers the true value of the mean.

2) U stands for the standardized normal variate : the value u_α is defined by :

$$P\,[U < u_\alpha] = \alpha$$

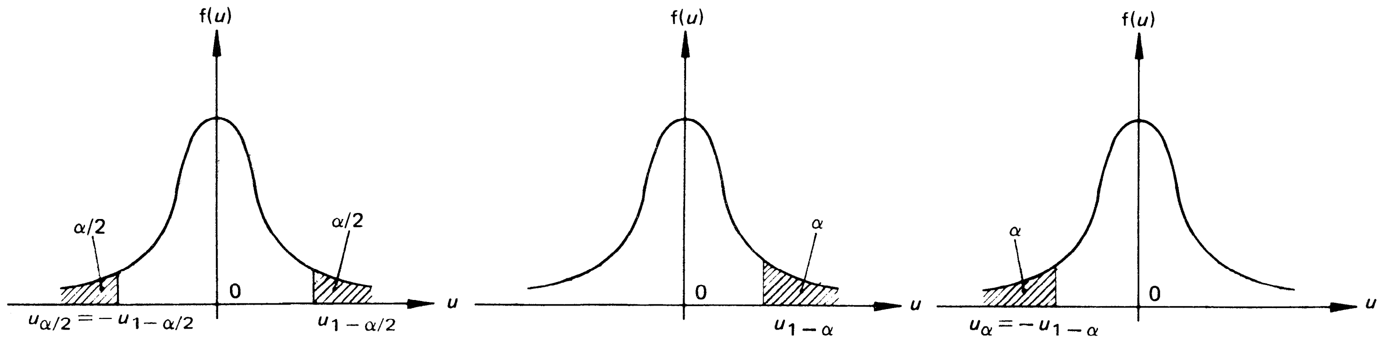
Since the distribution of U is symmetrical around zero, $u_\alpha = -\,u_{1-\alpha}$

We therefore have :

$$P\,[U > u_\alpha] = 1-\alpha$$

$$P\,[-u_{1-\alpha/2} < U < u_{1-\alpha/2}] = 1-\alpha$$

Probability density of U (standardized normal distribution)



Two-sided case

One-sided cases

3) $\sigma/\sqrt{n}$ is the standard deviation of the mean $\bar{x}$, in a sample of n observations.

4) For convenience in application, values of $u_{1-\alpha/2}/\sqrt{n}$ and $u_{1-\alpha}/\sqrt{n}$ are given in table I of annex B for $\alpha = 0{,}05$ and $\alpha = 0{,}01$.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE B′ — **Estimation of a mean** (variance unknown)

Technical characteristics of the population studied (5) . . . .

Technical characteristics of the sample items (5) . . . .

Discarded observations (6) . . . .

Statistical data	**Calculations**
Sample size : $n =$	$\bar{x} = \frac{\Sigma x}{n} =$
Sum of the observed values : $\Sigma x =$	$\frac{\Sigma (x - \bar{x})^2}{n-1} = \frac{\Sigma x^2 - (\Sigma x)^2/n}{n-1} =$
Sum of the squares of the observed values : $\Sigma x^2 =$	$\sigma^* = s = \sqrt{\frac{\Sigma (x - \bar{x})^2}{n-1}} =$
Degrees of freedom : $\nu = n - 1 =$	$[t_{1-\alpha}(\nu)/\sqrt{n}\,]\, s =$
Confidence level chosen (7) : $1 - \alpha =$	$[t_{1-\alpha/2}(\nu)/\sqrt{n}\,]\, s =$

Results

Estimation of the population mean m :

$$m^* = \bar{x} =$$

Two-sided confidence interval :

$$\bar{x} - [t_{1-\alpha/2}(\nu)/\sqrt{n}\,]\, s < m < \bar{x} + [t_{1-\alpha/2}(\nu)/\sqrt{n}\,]\, s$$

One-sided confidence intervals :

$$m < \bar{x} + [t_{1-\alpha}(\nu)/\sqrt{n}\,]\, s$$

or

$$m > \bar{x} - [t_{1-\alpha}(\nu)/\sqrt{n}\,]\, s$$

NOTE — The numbers (5), (6) and (7) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The confidence level $1-\alpha$ (see § 7 of the "General remarks") is the probability that the confidence interval covers the true value of the mean.

2) $t(\nu)$ stands for Student's variate with ν degrees of freedom; the value $t_\alpha(\nu)$ is defined by

$$P\,[t(\nu) < t_\alpha(\nu)] = \alpha$$

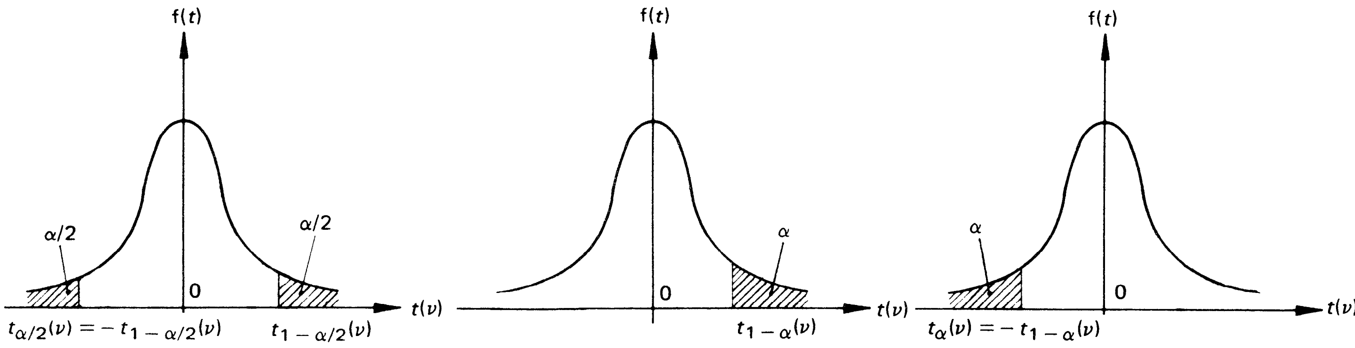
Since the distribution of $t(\nu)$ is symmetrical around zero, $t_\alpha(\nu) = -\,t_{1-\alpha}(\nu)$.

We therefore have :

$$P\,[t(\nu) > t_\alpha(\nu)] = 1-\alpha$$

$$P\,[-\,t_{1-\alpha/2}(\nu) < t(\nu) < t_{1-\alpha/2}(\nu)] = 1-\alpha$$

Probability density of Student's $t(\nu)$ with $\nu = n-1$ degrees of freedom



Two-sided case

One-sided cases

3) $\sigma^*/\sqrt{n}$ is the estimated standard deviation of the mean $\bar{x}$, in a sample of n observations.

4) For convenience in application, values of $t_{1-\alpha/2}(\nu)/\sqrt{n}$ and $t_{1-\alpha}(\nu)/\sqrt{n}$ are given in table IIb of annex B for $\alpha = 0{,}05$ and $\alpha = 0{,}01$.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE C — **Comparison of two means** (variances known)

Technical characteristics (5)	of population 1 . . . . of population 2	
Technical characteristics of the sample items taken (5)	in population 1 . . . . in population 2	
Discarded observations (6)	in sample 1 . . . . in sample 2	

Statistical data	First sample	Second sample	**Calculations**
			$\bar{x}_1 = \frac{\Sigma x_1}{n_1} =$
Size	$n_1 =$	$n_2 =$	
Sum of the observed values	$\Sigma x_1 =$	$\Sigma x_2 =$	$\bar{x}_2 = \frac{\Sigma x_2}{n_2} =$
Known values of the variances of the populations	$\sigma_1^2 =$	$\sigma_2^2 =$	$\sigma_d = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} =$
Significance level chosen (8) :			$u_{1-\alpha}\, \sigma_d =$
$\alpha =$			$u_{1-\alpha/2}\sigma_d =$

Results

Comparison of the two population means :

Two-sided case :

The hypothesis of the equality of the means (null hypothesis) is rejected if :

$$|\bar{x}_1 - \bar{x}_2| > u_{1-\alpha/2}\sigma_d$$

One-sided cases :

a) The hypothesis that the first mean is not smaller than the second (null hypothesis) is rejected if :

$$\bar{x}_1 < \bar{x}_2 - u_{1-\alpha}\sigma_d$$

b) The hypothesis that the first mean is not greater than the second (null hypothesis) is rejected if :

$$\bar{x}_1 > \bar{x}_2 + u_{1-\alpha}\sigma_d$$

NOTE — The numbers (5), (6) and (8) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The significance level α (see § 8 of the "General remarks") is the probability of rejecting the null hypothesis when this hypothesis is true.

2) U stands for the standardized normal variate : the value u_α is defined by :

$$P[U < u_\alpha] = \alpha$$

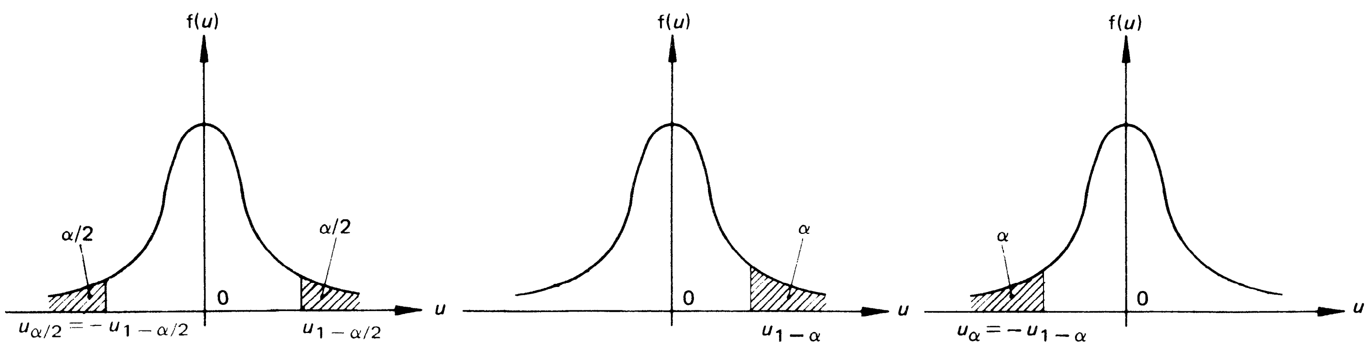
Since the distribution of U is symmetrical around zero, $u_\alpha = -u_{1-\alpha}$.

We therefore have :

$$P[U > u_\alpha] = 1 - \alpha$$

$$P[-u_{1-\alpha/2} < U < u_{1-\alpha/2}] = 1 - \alpha$$

Probability density of U (standardized normal distribution)



Two-sided case

One-sided cases

3) $\sigma_d = \sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$ is the standard deviation of the difference $d = \bar{x}_1 - \bar{x}_2$ of the means of the two samples of n_1 and n_2 observations respectively.

4) The values $u_{1-\alpha/2}$ and $u_{1-\alpha}$ must be read for $\alpha = 0{,}05$ and $\alpha = 0{,}01$ on the line $n = 1$ of table 1 of annex B.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE C′ – **Comparison of two means** (variances unknown, but may be assumed equal)

The hypothesis of the equality of the variances of the two populations can be tested as indicated in table G.

Technical characteristics (5)	of population 1	. . . .
	of population 2	. . . .
Technical characteristics of the sample items taken (5)	in population 1	. . . .
	in population 2	. . . .
Discarded observations (6)	in sample 1	. . . .
	in sample 2	. . . .

Statistical data

	First sample	Second sample
Size	$n_1 =$	$n_2 =$
Sum of the observed values	$\Sigma x_1 =$	$\Sigma x_2 =$
Sum of the squares of the observed values	$\Sigma x_1^2 =$	$\Sigma x_2^2 =$
Degrees of freedom	$\nu = n_1 + n_2 - 2 =$	

Significance level chosen (8) :

$\alpha =$

Calculations

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} =$$

$$\bar{x}_2 = \frac{\Sigma x_2}{n_2} =$$

$$\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2 =$$

$$\Sigma x_1^2 + \Sigma x_2^2 - \frac{1}{n_1} (\Sigma x_1)^2 - \frac{1}{n_2} (\Sigma x_2)^2 =$$

$$\sigma_d^* = s_d = \sqrt{\frac{n_1 + n_2}{n_1 n_2} \; \frac{\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}} =$$

$$t_{1-\alpha}(\nu)\, s_d =$$

$$t_{1-\alpha/2}(\nu)\, s_d =$$

Results

Comparison of the two populations means :

Two-sided case :

The hypothesis of the equality of the means (null hypothesis) is rejected if :

$$|\bar{x}_1 - \bar{x}_2| > t_{1-\alpha/2}(\nu)\, s_d$$

One-sided cases :

a) The hypothesis that the first mean is not smaller than the second (null hypothesis) is rejected if :

$$\bar{x}_1 < \bar{x}_2 - t_{1-\alpha}(\nu)\, s_d$$

b) The hypothesis that the first mean is not greater than the second (null hypothesis) is rejected if :

$$\bar{x}_1 > \bar{x}_2 + t_{1-\alpha}(\nu)\, s_d$$

NOTE – The numbers (5), (6) and (8) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The significance level α (see § 8 of the "General remarks") is the probability of rejecting the null hypothesis when this hypothesis is true.

2) $t(\nu)$ stands for Student's variate with $\nu = n_1 + n_2 - 2$ degrees of freedom; the value $t_\alpha(\nu)$ is defined by :

$$P\,[t(\nu) < t_\alpha(\nu)] = \alpha$$

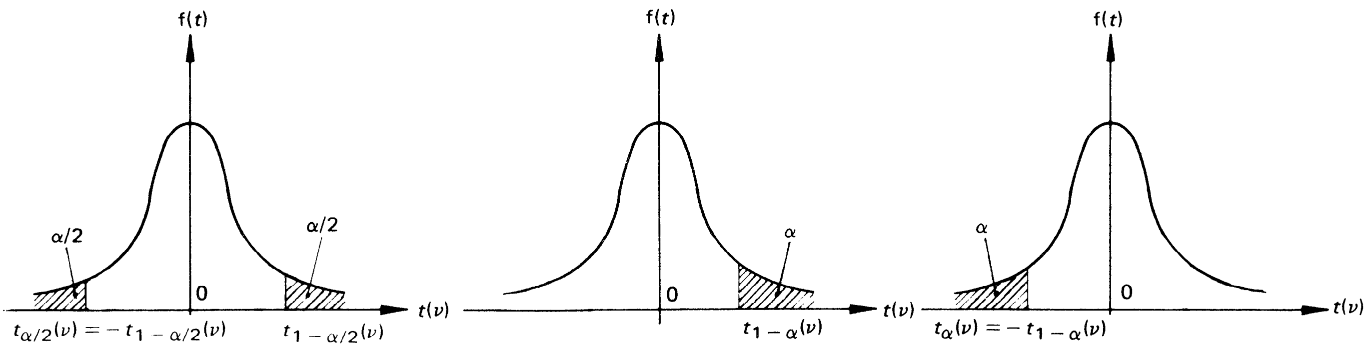
Since the distribution of $t(\nu)$ is symmetrical around zero, $t_\alpha(\nu) = -t_{1-\alpha}(\nu)$.

We therefore have :

$$P\,[t(\nu) > t_\alpha(\nu)] = 1 - \alpha$$

$$P\,[-t_{1-\alpha/2}(\nu) < t(\nu) < t_{1-\alpha/2}(\nu)] = 1 - \alpha$$

Probability density of Student's $t(\nu)$ with $\nu = n_1 + n_2 - 2$ degrees of freedom



Two-sided case

One-sided cases

3) σ_d^* is the estimated standard deviation of the difference $d = \bar{x}_1 - \bar{x}_2$ of the means of the two samples of n_1 and n_2 observations respectively.

4) The values $t_{1-\alpha/2}(\nu)$ and $t_{1-\alpha}(\nu)$ are given in table IIa of annex B for $\alpha = 0{,}05$ and $\alpha = 0{,}01$.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE D – **Estimation of the difference of two means** (variances known)

Technical characteristics (5)	of population 1 . . . . of population 2	
Technical characteristics of the sample items taken (5)	in population 1 . . . . in population 2	
Discarded observations (6)	in sample 1 . . . . in sample 2	

Statistical data	First sample	Second sample	**Calculations**
Size	$n_1 =$	$n_2 =$	$\bar{x}_1 = \frac{\Sigma x_1}{n_1} =$ $\bar{x}_2 = \frac{\Sigma x_2}{n_2} =$
Sum of the observed values	$\Sigma x_1 =$	$\Sigma x_2 =$	
Known values of the variances of the populations	$\sigma_1^2 =$	$\sigma_2^2 =$	$\sigma_d = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} =$ $u_{1-\alpha}\sigma_d =$
Confidence level chosen (7) : $1 - \alpha =$			$u_{1-\alpha/2}\sigma_d =$

Results

Estimation of the difference of the two populations means m_1 and m_2 :

$$(m_1 - m_2)^* = \bar{x}_1 - \bar{x}_2 =$$

Two-sided confidence interval :

$$(\bar{x}_1 - \bar{x}_2) - u_{1-\alpha/2}\sigma_d < m_1 - m_2 < (\bar{x}_1 - \bar{x}_2) + u_{1-\alpha/2}\sigma_d$$

One-sided confidence intervals :

$$m_1 - m_2 < (\bar{x}_1 - \bar{x}_2) + u_{1-\alpha}\sigma_d$$

or

$$m_1 - m_2 > (\bar{x}_1 - \bar{x}_2) - u_{1-\alpha}\sigma_d$$

NOTE – The numbers (5), (6) and (7) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The confidence level $1 - \alpha$ (see § 7 of the "General remarks") is the probability that the calculated confidence interval covers the true value of the difference between the means.

2) U stands for the standardized normal variate : the value u_α is defined by :

$$P[U < u_\alpha] = \alpha$$

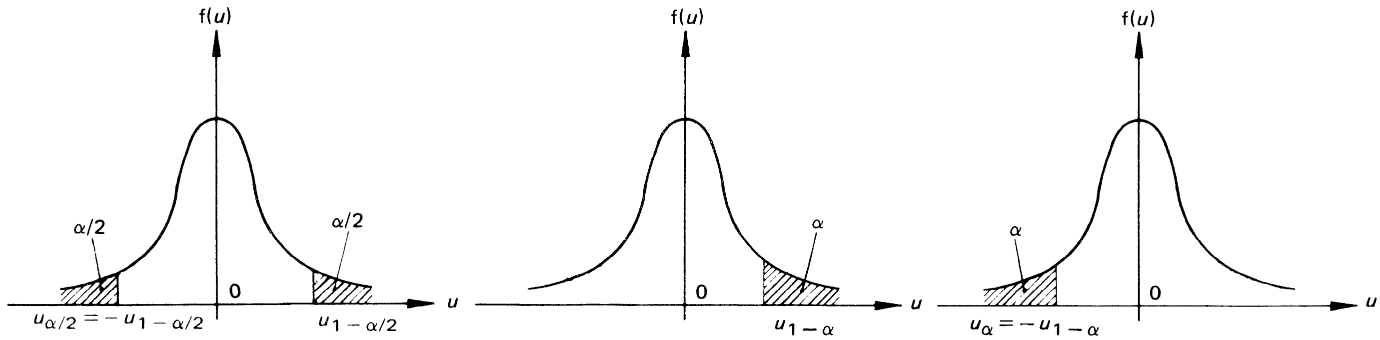
Since the distribution of U is symmetrical around zero, $u_\alpha = -u_{1-\alpha}$.

We therefore have :

$$P[U > u_\alpha] = 1 - \alpha$$

$$P[-u_{1-\alpha/2} < U < u_{1-\alpha/2}] = 1 - \alpha$$

Probability density of U (standardized normal distribution)



Two-sided case

One-sided case

3) $\sigma_d = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ is the standard deviation of the difference $d = \bar{x}_1 - \bar{x}_2$ between the means of the two samples of n_1 and n_2 observations respectively.

4) The values $u_{1-\alpha/2}$ and $u_{1-\alpha}$ must be read for $\alpha = 0{,}05$ and $\alpha = 0{,}01$ on the line $n = \infty$ of table 1 of annex B.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE D′ — **Estimation of the difference of two means** (variances unknown, but may be assumed equal)

The hypothesis of the equality of the variances of the two populations can be tested as indicated in table G.

Technical characteristics (5)	of population 1	
	of population 2	
Technical characteristics of the sample items taken (5)	in population 1	
	in population 2	
Discarded observations (6)	in sample 1	
	in sample 2	

Statistical data	First sample	Second sample	**Calculations**
			$\bar{x}_1 = \frac{\Sigma x_1}{n_1} =$
Size	$n_1 =$	$n_2 =$	$\bar{x}_2 = \frac{\Sigma x_2}{n_2} =$
Sum of the observed values	$\Sigma x_1 =$	$\Sigma x_2 =$	$\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2 =$
Sum of the squares of the observed values	$\Sigma x_1^2 =$	$\Sigma x_2^2 =$	$\Sigma x_1^2 + \Sigma x_2^2 - \frac{1}{n_1} (\Sigma x_1)^2 - \frac{1}{n_2} (\Sigma x_2)^2 =$
Degrees of freedom	$\nu = n_1 + n_2 - 2 =$		$\sigma_d^* = s_d = \sqrt{\frac{n_1 + n_2}{n_1 n_2} \frac{\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}}$
Confidence level chosen (7) :			$t_{1-\alpha}(\nu)\, s_d =$
$1 - \alpha =$			$t_{1-\alpha/2}(\nu)\, s_d =$

Results

Estimation of the difference of the two populations means m_1 and m_2 :

$$(m_1 - m_2)^* = \bar{x}_1 - \bar{x}_2 =$$

Two-sided confidence interval :

$$(\bar{x}_1 - \bar{x}_2) - t_{1-\alpha/2}(\nu)\, s_d < m_1 - m_2 < (\bar{x}_1 - \bar{x}_2) + t_{1-\alpha/2}(\nu)\, s_d$$

One-sided confidence intervals :

$$m_1 - m_2 < (\bar{x}_1 - \bar{x}_2) + t_{1-\alpha}(\nu)\, s_d$$

or

$$m_1 - m_2 > (\bar{x}_1 - \bar{x}_2) - t_{1-\alpha}(\nu)\, s_d$$

NOTE — The numbers (5), (6) and (7) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The confidence level $1-\alpha$ (see § 7 of the "General remarks") is the probability that the calculated confidence interval covers the true value of the difference between the means.

2) $t(\nu)$ stands for Student's variate with $\nu = n_1 + n_2 - 2$ degrees of freedom; the value $t_\alpha(\nu)$ is defined by

$$P\,[t(\nu) < t_\alpha(\nu)] = \alpha$$

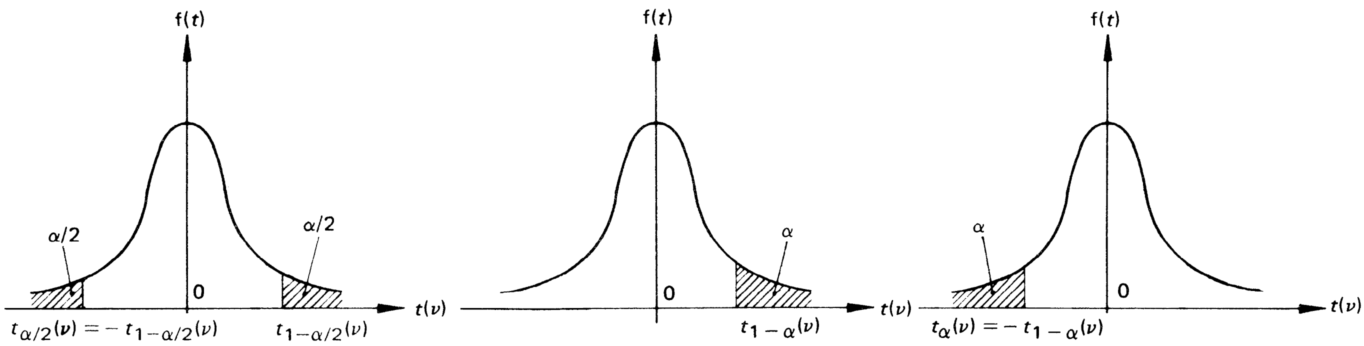
Since the distribution of $t(\nu)$ is symmetrical around zero, $t_\alpha(\nu) = -t_{1-\alpha}(\nu)$.

We therefore have :

$$P\,[t(\nu) > t_\alpha(\nu)] = 1-\alpha$$

$$P\,[-t_{1-\alpha/2}(\nu) < t(\nu) < t_{1-\alpha/2}(\nu)] = 1-\alpha$$

Probability density of Student's $t(\nu)$ with $\nu = n_1 + n_2 - 2$ degrees of freedom



Two-sided case — One-sided cases

3) σ_d^* is the estimated standard deviation of the difference $d = \bar{x}_1 - \bar{x}_2$ between the means of the two samples of n_1 and n_2 observations respectively.

4) The values $t_{1-\alpha/2}(\nu)$ and $t_{1-\alpha}(\nu)$ are given in table IIa of annex B for $\alpha = 0{,}05$ and $\alpha = 0{,}01$.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE E — **Comparison of a variance or of a standard deviation with a given value**

Technical characteristics of the population studied (5) . . . .

Technical characteristics of the sample elements (5) . . . .

Discarded observations (6) . . . .

Statistical data	**Calculations**
Sample size : $n =$	
Sum of the observed values : $\Sigma x =$	$\Sigma (x - \bar{x})^2 = \Sigma x^2 - \dfrac{(\Sigma x)^2}{n} =$
Sum of the squares of the observed values : $\Sigma x^2 =$	$\dfrac{\Sigma (x - \bar{x})^2}{\sigma_0^2} =$
Given value : $\sigma_0^2 =$	$\chi^2_{\alpha}(\nu) =$
Degrees of freedom : $\nu = n - 1 =$	$\chi^2_{1-\alpha}(\nu) =$
Significance level chosen (8) : $\alpha =$	$\chi^2_{\alpha/2}(\nu) =$ $\chi^2_{1-\alpha/2}(\nu) =$

Results

Comparison of the population variance with the given value σ_0^2 :

Two-sided case :

The hypothesis that the population variance is equal to the given value (null hypothesis) is rejected if :

$$\frac{\Sigma (x - \bar{x})^2}{\sigma_0^2} < \chi^2_{\alpha/2}(\nu) \text{ or } \frac{\Sigma (x - \bar{x})^2}{\sigma_0^2} > \chi^2_{1-\alpha/2}(\nu)$$

One-sided cases :

a) The hypothesis that the population variance is not larger than the given value (null hypothesis) is rejected if :

$$\frac{\Sigma (x - \bar{x})^2}{\sigma_0^2} > \chi^2_{1-\alpha}(\nu)$$

b) The hypothesis that the population variance is not smaller than the given value (null hypothesis) is rejected if :

$$\frac{\Sigma (x - \bar{x})^2}{\sigma_0^2} < \chi^2_{\alpha}(\nu)$$

NOTE — The numbers (5), (6) and (8) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The significance level α (see § 8 of the "General remarks") is the probability of rejecting the null hypothesis when this hypothesis is true.

2) $\chi^2(\nu)$ stands for the χ^2 variate with ν degrees of freedom; the value $\chi^2_\alpha(\nu)$ is defined by

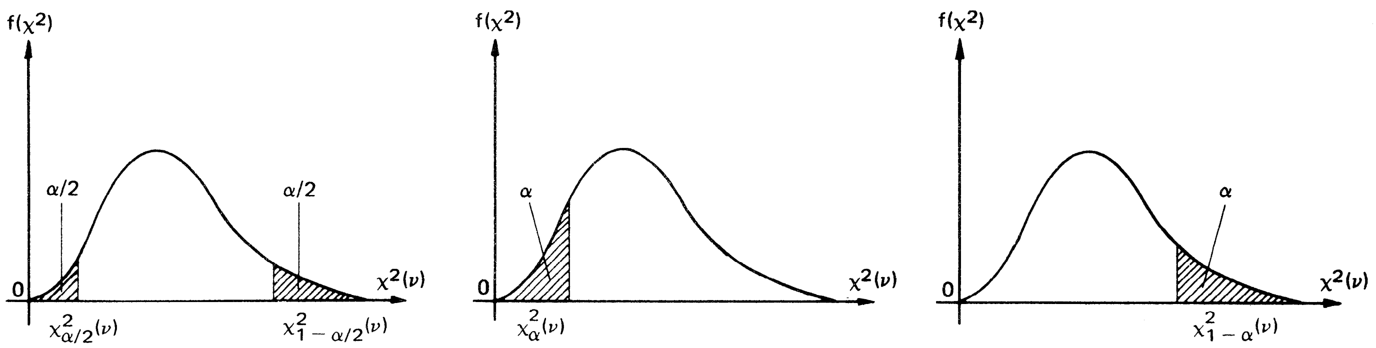
$$P\left[\chi^2(\nu) < \chi^2_\alpha(\nu)\right] = \alpha$$

We therefore have :

$$P\left[\chi^2(\nu) > \chi^2_\alpha(\nu)\right] = 1 - \alpha$$

$$P\left[\chi^2_{\alpha/2}(\nu) < \chi^2(\nu) < \chi^2_{1-\alpha/2}(\nu)\right] = 1 - \alpha$$

Probability density of $\chi^2(\nu)$ with $\nu = n - 1$ degrees of freedom



Two-sided case

One-sided cases

3) The values $\chi^2_\alpha(\nu)$, $\chi^2_{1-\alpha}(\nu)$, $\chi^2_{\alpha/2}(\nu)$ and $\chi^2_{1-\alpha/2}(\nu)$ are given in table III of annex B for $\alpha = 0{,}05$ and $\alpha = 0{,}01$.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE F — **Estimation of a variance or of a standard deviation**

Technical characteristics of the population studied (5) . . . .

Technical characteristics of the sample items (5) . . . .

Discarded observations (6) . . . .

Statistical data	**Calculations**
Sample size : $n =$	$\Sigma (x - \bar{x})^2 = \Sigma x^2 - \frac{(\Sigma x)^2}{n} =$
Sum of the observed values : $\Sigma x =$	$s^2 = \frac{\Sigma (x - \bar{x})^2}{n - 1} =$
Sum of the squares of the observed values : $\Sigma x^2 =$	$\frac{\Sigma (x - \bar{x})^2}{\chi^2_{\alpha}(\nu)} =$
Degrees of freedom : $\nu = n - 1 =$	$\frac{\Sigma (x - \bar{x})^2}{\chi^2_{1-\alpha}(\nu)} =$
Confidence level chosen (7) : $1 - \alpha =$	$\frac{\Sigma (x - \bar{x})^2}{\chi^2_{\alpha/2}(\nu)} =$
	$\frac{\Sigma (x - \bar{x})^2}{\chi^2_{1-\alpha/2}(\nu)} =$

Results

Estimation of the population variance σ^2 :

$$(\sigma^2)^* = s^2 =$$

Two-sided confidence interval[1] :

$$\frac{\Sigma (x - \bar{x})^2}{\chi^2_{1-\alpha/2}(\nu)} < \sigma^2 < \frac{\Sigma (x - \bar{x})^2}{\chi^2_{\alpha/2}(\nu)}$$

One-sided confidence intervals[1] :

$$\sigma^2 < \frac{\Sigma (x - \bar{x})^2}{\chi^2_{\alpha}(\nu)}$$

or

$$\sigma^2 > \frac{\Sigma (x - \bar{x})^2}{\chi^2_{1-\alpha}(\nu)}$$

1) The limits of the confidence intervals of the standard deviation σ are the square roots of the limits of the confidence intervals of the variance σ^2.

NOTE — The numbers (5), (6) and (7) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The confidence level $1 - \alpha$ (see § 7 of the "General remarks") is the probability that the calculated confidence interval covers the true value of the variance.

2) $\chi^2(\nu)$ stands for the χ^2 variate with $\nu = n - 1$ degrees of freedom; the value $\chi^2_\alpha(\nu)$ is defined by

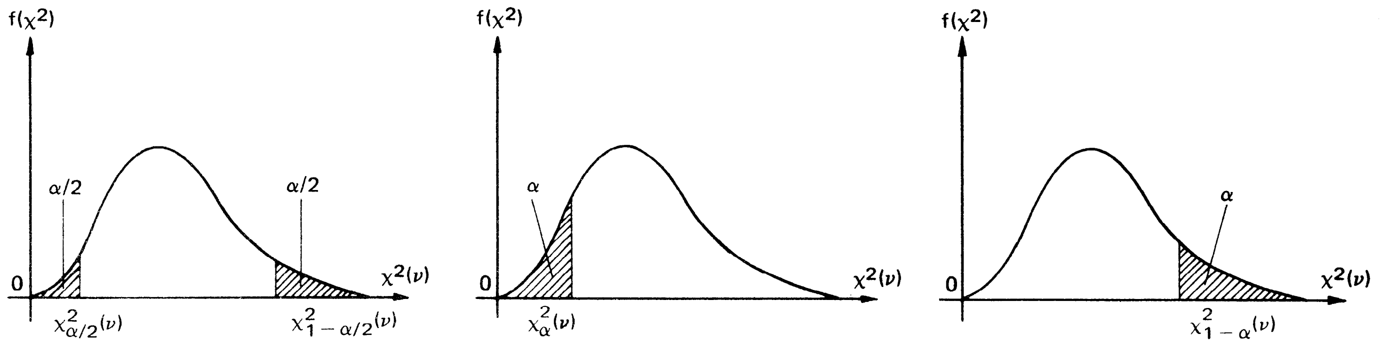
$$P\left[\chi^2(\nu) < \chi^2_\alpha(\nu)\right] = \alpha$$

We therefore have :

$$P\left[\chi^2(\nu) > \chi^2_\alpha(\nu)\right] = 1 - \alpha$$

$$P\left[\chi^2_{\alpha/2}(\nu) < \chi^2(\nu) < \chi^2_{1-\alpha/2}(\nu)\right] = 1 - \alpha$$

Probability density of $\chi^2(\nu)$ with $\nu = n - 1$ degrees of freedom



Two-sided case

One-sided cases

3) The values $\chi^2_\alpha(\nu)$, $\chi^2_{1-\alpha}(\nu)$, $\chi^2_{\alpha/2}(\nu)$ and $\chi^2_{1-\alpha/2}(\nu)$ are given in table III of annex B for $\alpha = 0{,}05$ and $\alpha = 0{,}01$.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE G – **Comparison of two variances or of two standard deviations**

Technical characteristics (5)	of population 1	. . . .
	of population 2	. . . .
Technical characteristics of the sample items taken (5)	in population 1	. . . .
	in population 2	. . . .
Discarded observations (6)	in sample 1	. . . .
	in sample 2	. . . .

Statistical data

	First sample	Second sample
Size	$n_1 =$	$n_2 =$
Sum of the observed values	$\Sigma x_1 =$	$\Sigma x_2 =$
Sum of the squares of the observed values	$\Sigma x_1^2 =$	$\Sigma x_2^2 =$
Degrees of freedom	$\nu_1 = n_1 - 1$	$\nu_2 = n_2 - 1$

Significance level chosen (8) :

$\alpha =$

Calculations

$$\Sigma (x_1 - \bar{x}_1)^2 = \Sigma x_1^2 - \frac{(\Sigma x_1)^2}{n_1} =$$

$$\Sigma (x_2 - \bar{x}_2)^2 = \Sigma x_2^2 - \frac{(\Sigma x_2)^2}{n_2} =$$

$$s_1^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2}{n_1 - 1} =$$

$$s_2^2 = \frac{\Sigma (x_2 - \bar{x}_2)^2}{n_1 - 1} =$$

$F_{1-\alpha}(\nu_1, \nu_2) =$	$F_{1-\alpha/2}(\nu_1, \nu_2) =$
$\dfrac{1}{F_{1-\alpha}(\nu_2, \nu_1)} =$	$\dfrac{1}{F_{1-\alpha/2}(\nu_2, \nu_1)} =$

Results

Comparison of the population variances :

Two-sided case :

The hypothesis of the equality of the variances (null hypothesis) is rejected if :

$$\frac{s_1^2}{s_2^2} < \frac{1}{F_{1-\alpha/2}(\nu_2, \nu_1)} \quad \text{or} \quad \frac{s_1^2}{s_2^2} > F_{1-\alpha/2}(\nu_1, \nu_2)$$

One-sided cases :

a) The hypothesis that the first variance is not greater than the second (null hypothesis) is rejected if :

$$\frac{s_1^2}{s_2^2} > F_{1-\alpha}(\nu_1, \nu_2)$$

b) The hypothesis that the first variance is not smaller than the second (null hypothesis) is rejected if :

$$\frac{s_1^2}{s_2^2} < \frac{1}{F_{1-\alpha}(\nu_2, \nu_1)}$$

NOTE – The numbers (5), (6) and (8) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The significance level α (see § 8 of the "General remarks") is the probability of rejecting the null hypothesis when this hypothesis is true.

2) $F(\nu_1, \nu_2)$ stands for the variance ratio with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom; the value $F_\alpha(\nu_1, \nu_2)$ is defined by :

$$P\,[F(\nu_1, \nu_2) < F_\alpha(\nu_1, \nu_2)] = \alpha$$

We therefore have :

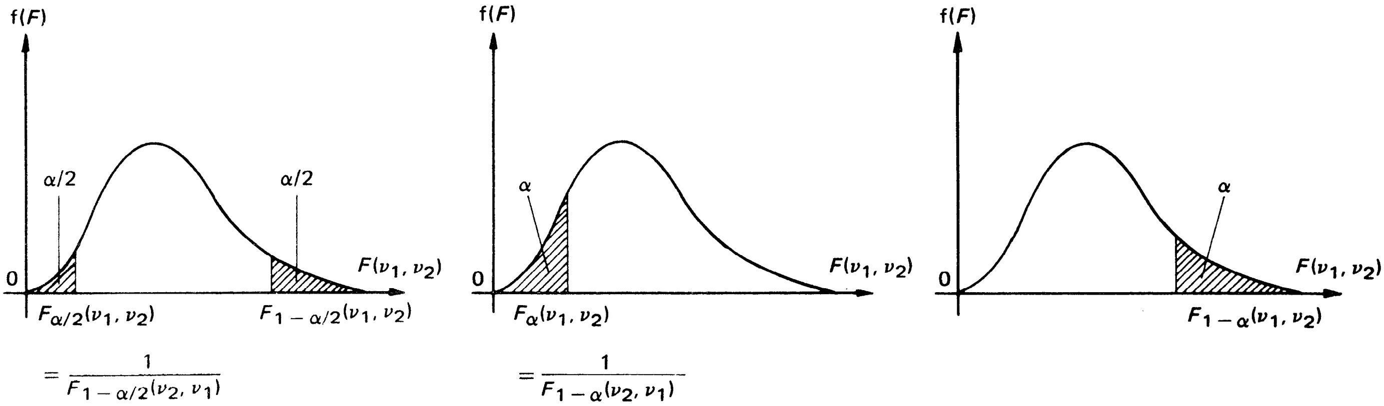
$$P\,[F(\nu_1, \nu_2) > F_\alpha(\nu_1, \nu_2)] = 1 - \alpha$$

$$P\,[F_{\alpha/2}(\nu_1, \nu_2) < F(\nu_1, \nu_2) < F_{1-\alpha/2}(\nu_1, \nu_2)] = 1 - \alpha$$

We also have :

$$F_\alpha(\nu_1, \nu_2) = \frac{1}{F_{1-\alpha}(\nu_2, \nu_1)}$$

Probability density of $F(\nu_1, \nu_2)$ with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom



Two-sided case

One-sided cases

3) The values $F_{1-\alpha}$ and $F_{1-\alpha/2}$ are given in table IV of annex B as functions of the numbers of degrees of freedom, for $\alpha = 0{,}05$ and $\alpha = 0{,}01$. The values F_α and $F_{\alpha/2}$ may be derived as indicated above from the values $F_{1-\alpha}$ and $F_{1-\alpha/2}$.

EXAMPLE : see section two, "Explanatory notes and examples".

TABLE H — **Estimation of the ratio of two variances or of two standard deviations**

Technical characteristics (5)	of population 1	. . . .
	of population 2	. . . .
Technical characteristics of the sample items taken (5)	in population 1	. . . .
	in population 2	. . . .
Discarded observations (6)	in sample 1	. . . .
	in sample 2	. . . .

Statistical data

	First sample	Second sample
Size	$n_1 =$	$n_2 =$
Sum of the observed values	$\Sigma x_1 =$	$\Sigma x_2 =$
Sum of the squares of the observed values	$\Sigma x_1^2 =$	$\Sigma x_2^2 =$
Degrees of freedom	$\nu_1 = n_1 - 1$	$\nu_2 = n_2 - 1$

Confidence level chosen (7) :

$1 - \alpha =$

Calculations

$$\Sigma (x_1 - \bar{x}_1)^2 = \Sigma x_1^2 - \frac{(\Sigma x_1)^2}{n_1} =$$

$$\Sigma (x_2 - \bar{x}_2)^2 = \Sigma x_2^2 - \frac{(\Sigma x_2)^2}{n_2} =$$

$$s_1^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2}{n_2 - 1} =$$

$$s_2^2 = \frac{\Sigma (x_2 - \bar{x}_2)^2}{n_2 - 1} =$$

$$F_{1-\alpha}(\nu_2, \nu_1) \frac{s_1^2}{s_2^2} = \qquad F_{1-\alpha/2}(\nu_2, \nu_1) \frac{s_1^2}{s_2^2} =$$

$$F_{1-\alpha}(\nu_1, \nu_2) \frac{s_1^2}{s_2^2} = \qquad F_{1-\alpha/2}(\nu_1, \nu_2) \frac{s_1^2}{s_2^2} =$$

Results

Estimation of the ratio of the two population variances σ_1^2 and σ_2^2 :

$$\left(\frac{\sigma_1^2}{\sigma_2^2}\right)^* = \frac{s_1^2}{s_2^2} = \frac{\Sigma (x_1 - \bar{x}_1)^2/(n_1 - 1)}{\Sigma (x_2 - \bar{x}_2)^2/(n_1 - 1)}$$

Two-sided confidence interval[1] :

$$\frac{1}{F_{1-\alpha/2}(\nu_1, \nu_2)} \frac{s_1^2}{s_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < F_{1-\alpha/2}(\nu_2, \nu_1) \frac{s_1^2}{s_2^2}$$

One-sided confidence intervals[1] :

$$\frac{\sigma_1^2}{\sigma_2^2} < F_{1-\alpha}(\nu_2, \nu_1) \frac{s_1^2}{s_2^2} \quad \text{or} \quad \frac{\sigma_1^2}{\sigma_2^2} > \frac{1}{F_{1-\alpha}(\nu_1, \nu_2)} \frac{s_1^2}{s_2^2}$$

1) The limits of the confidence intervals of the ratio of the standard deviations σ_1 and σ_2 are the square roots of the limits of the confidence intervals of the ratio of the variances σ_1^2 and σ_2^2.

NOTE — The numbers (5), (6) and (7) refer to the corresponding paragraphs of the "General remarks".

Comments

1) The confidence level $1-\alpha$ (see § 7 of the "General remarks") is the probability that the calculated confidence interval covers the true value of the ratio of the two variances.

2) $F(\nu_1, \nu_2)$ stands for the variance ratio with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom; the value $F_\alpha(\nu_1, \nu_2)$ is defined by :

$$P\left[F(\nu_1, \nu_2) < F_\alpha(\nu_1, \nu_2)\right] = \alpha$$

We therefore have :

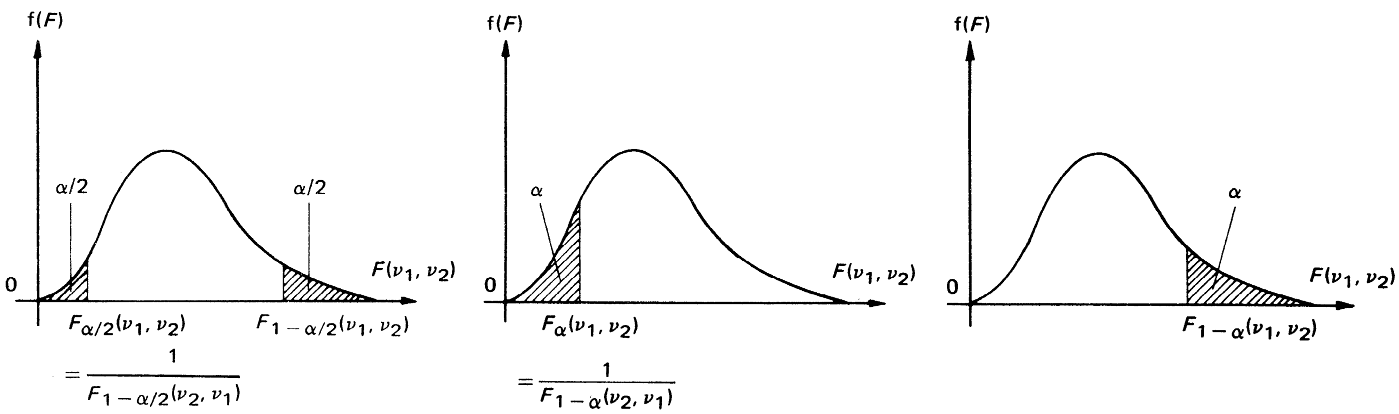
$$P\left[F(\nu_1, \nu_2) > F_\alpha(\nu_1, \nu_2)\right] = 1-\alpha$$

$$P\left[F_{\alpha/2}(\nu_1, \nu_2) < F(\nu_1, \nu_2) < F_{1-\alpha/2}(\nu_1, \nu_2)\right] = 1-\alpha$$

We also have :

$$F_\alpha(\nu_1, \nu_2) = \frac{1}{F_{1-\alpha}(\nu_2, \nu_1)}$$

Probability density of $F(\nu_1, \nu_2)$, with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom



Two-sided case

One-sided cases

3) The values $F_{1-\alpha}$ and $F_{1-\alpha/2}$ are given in table IV of annex B as functions of the numbers of degrees of freedom, for $\alpha = 0{,}05$ and $\alpha = 0{,}01$. The values F_α and $F_{\alpha/2}$ may be derived from the values $F_{1-\alpha}$ and $F_{1-\alpha/2}$ as indicated above.

EXAMPLE : see section two, "Explanatory notes and examples".

SECTION TWO : EXPLANATORY NOTES AND EXAMPLES

INTRODUCTORY REMARKS

1) The tables given in section one of this International Standard set out formally twelve different procedures which can be applied to data observed in samples in order to help answer a variety of questions regarding the larger population or populations from which it is supposed that the sample(s) has (have) been randomly drawn. To add to the understanding of the more formal presentation given in tables A to H, the procedures will now be illustrated on numerical data consisting of measurements of breaking load for two samples of yarn. The most important characteristics of the samples are printed beside the observations in table X.

The unit in which the numerical data and the calculations results are expressed is the newton.

TABLE X — Breaking load of yarn (in newtons)

(For the meanings of the symbols, see, for instance, table G)

Yarn 1	Yarn 2
2,297	2,286
2,582	2,327
1,949	2,388
2,362	3,172
2,040	3,158
2,133	2,751
1,855	2,222
1,986	2,367
1,642	2,247
2,915	2,512
	2,104
	2,707

Sample sizes :

$n_1 = 10$ $n_2 = 12$

Sum of observed values Σx :

21,761 30,241

Mean values :

$\bar{x}_1 = 2,176$ $\bar{x}_2 = 2,520$

Sum of squares of observed values, Σx^2 :

48,610 477 77,599 609

Sum of squares of differences about means, $\Sigma(x - \bar{x})^2$:

1,256 365 1,389 769

Estimates of variance :

$s_1^2 = 0,139\ 60$ $s_2^2 = 0,126\ 34$

2) It is not suggested that answers to the whole set of questions would ever be required in a given investigation, but to simplify the presentation it is convenient to use the same illustrative material in each case. As a result it seems only necessary to illustrate numerically the complete formal presentation of the twelve tables in two cases: the single-sample case of table A and the two-sample case of table C.

In general the question or questions to be asked will be decided upon before the data are analyzed; indeed, it is best that they should determine the way in which the data are collected. However, a plot of the observations which are to be used in the examples will illustrate the kind of question which may be of interest. Some of these are as follows :

Allowing for chance sampling fluctuations, are the means or the standard deviations in the two samples consistent with the hypothesis that the two population means and/or standard deviations are identical?

If they are not identical, by how much may they differ?

The procedures set out in tables A to H give an objective backing, in terms of probability statements, to answers which may be suggested more tentatively by inspection of plots such as these.

3) Since the procedures to be followed depend on the assumption that the populations sampled are approximately represented by the normal density function, which in standardized form has the equation

$$f(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$$

as a start it is usually desirable to make a rough examination of this assumption, unless of course an adequate assurance of normality has been established from past examination of similar data. When the number of data is not very large, this examination may be made graphically used one of several alternative methods, two of which will be described here. Both involve arranging the observations in ascending order of magnitude[1], so that in a sample of n observations, x_i

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$$

1) With quite simple modification, the observations could alternatively be arranged in descending order of magnitude, i.e. $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$.

In the case of the second of the two samples of yarn given in table X, the twelve ordered observations are :

2,104 – 2,222 – 2,247 – 2,286 – 2,327 – 2,367 – 2,388 – 2,512 – 2,707 – 2,751 – 3,158 – 3,172

These ordered observations are termed the "order statistics of the sample", and in either method will be used as ordinates in the diagram to be plotted. The two methods differ in the abscissae used; in one, a), the expected values of the normal order statistics, are taken; in the second, b), the plotting is done on so-called "normal probability paper" and the chosen abscissa is the expected value of the cumulative probability associated with the order statistic.

a) **Use of expected values of normal order statistics, say $\xi(i|n)$**

For random samples of size n from a standardized normal distribution (i.e. with mean zero and unit standard deviation), these expected values, $\xi(i|n)$ are given in table V of annex B for $n = 2(1)50$, $i = 1, 2, ..., n/2$ for n even and $i = 1, 2, ..., (n+1)/2$ for n odd. H.L. Harter tables[1] give values of $\xi(i|n)$ for $n = 1(1)100$ and afterwards for rather wider intervals up to $n = 400$. The remaining values are obtained by giving negative signs to the values tabled, i.e. the expected order statistics for $i = n, n-1, n-2, ...,$ are those for $i = 1, 2, 3, ...,$ with signs reversed. If the twelve observed order statistics are plotted as ordinates against the corresponding expected values $\xi(i|n)$, $i = 1, 2, ... 12$, the result is the diagram shown in figure 2.

If the population distribution is strictly normal, the plotted points should only diverge from a straight line through chance sampling fluctuations. The slope of the line provides an estimate of the population standard deviation. This straight line gives an approximate estimation of the population mean (ordinate 2,52 of the abscissa point 0,0 of the straight line) and of its standard deviation (slope of the straight line, let for example 0,355 = the difference of ordinates between the two points of abscissa 1 and 0 of the straight line).

b) **Use of normal probability paper**

It is necessary to preface the description of this procedure with a few words about the nature of this paper, which may usually be obtained from any firm selling ruled papers having a variety of scales of grid.

If X is a random variate from a population having mean $= m$, standard deviation $= \sigma$, and if $U = (X - m)/\sigma$, it is clear that if we have n values of x_i, and plot x_i as ordinate against u_i as abscissa, the points (u_i, x_i) will fall on a straight line which will have slope σ and will pass through the point with co-ordinates $(0, m)$. If the population sampled is normal having a density function $F(u)$ as defined above, the uniform abscissa-scale, u, may be replaced by the probability scale, $P(u)$, where

$$P(u_i) = \int_{-\infty}^{u_i} e^{-u^2/2} \, du/\sqrt{2\pi}$$

The following table indicates certain corresponding values of 100 P and u.

100 P	u
0,1	− 3,090
0,5	− 2,576
1,0	− 2,326
2,5	− 1,960
5,0	− 1,645
10,0	− 1,282
20,0	− 0,842
25,0	− 0,674
30,0	− 0,524
40,0	− 0,253
50,0	0,000
60,0	0,253
70,0	0,524
75,0	0,674
80,0	0,842
90,0	1,282
95,0	1,645
97,5	1,960
99,0	2,326
99,5	2,576
99,9	3,090

Figure 3 shows a uniformly spaced vertical set of rules for x, while the horizontal rules are drawn against the scale of $P(u)$, rather than the uniformly spaced scale u. In the standard form of normal probability paper the scale u is, in fact, omitted.

In practice, of course, the population values of m and σ will generally be unknown so that neither the u_i or $P(u_i)$ corresponding to x_i can be determined. It is, however, known that if repeated random samples of n observations are drawn from a normal population and the individual observations in each sample arranged in ascending order of magnitude, x_i being the ith order statistic, then whatever be m and σ, the average or expected value of $P(x_i)$ is equal to $i/(n+1)$, that is it lies at a fixed point on the P-scale.

Given a single sample of size n, the graphical test for departure from normality, based on the use of normal probability paper, consists therefore in

a) assigning to the vertical x-grid a suitable scale according to the observed range of values of x in the sample;

b) plotting the ith normal order statistic x_i as ordinate against $P_i = i/(n+1)$ as abscissa.

1) Taken from H.L. Harter, *Order Statistics and their Use in Testing and Estimations,* Volume 2.

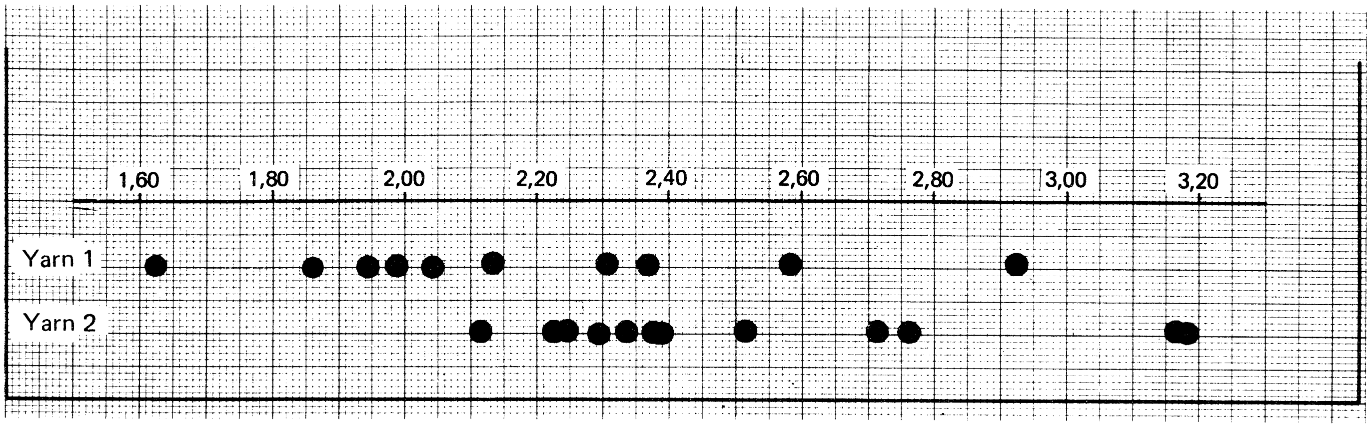


FIGURE 1 – **Breaking load of yarn in samples**

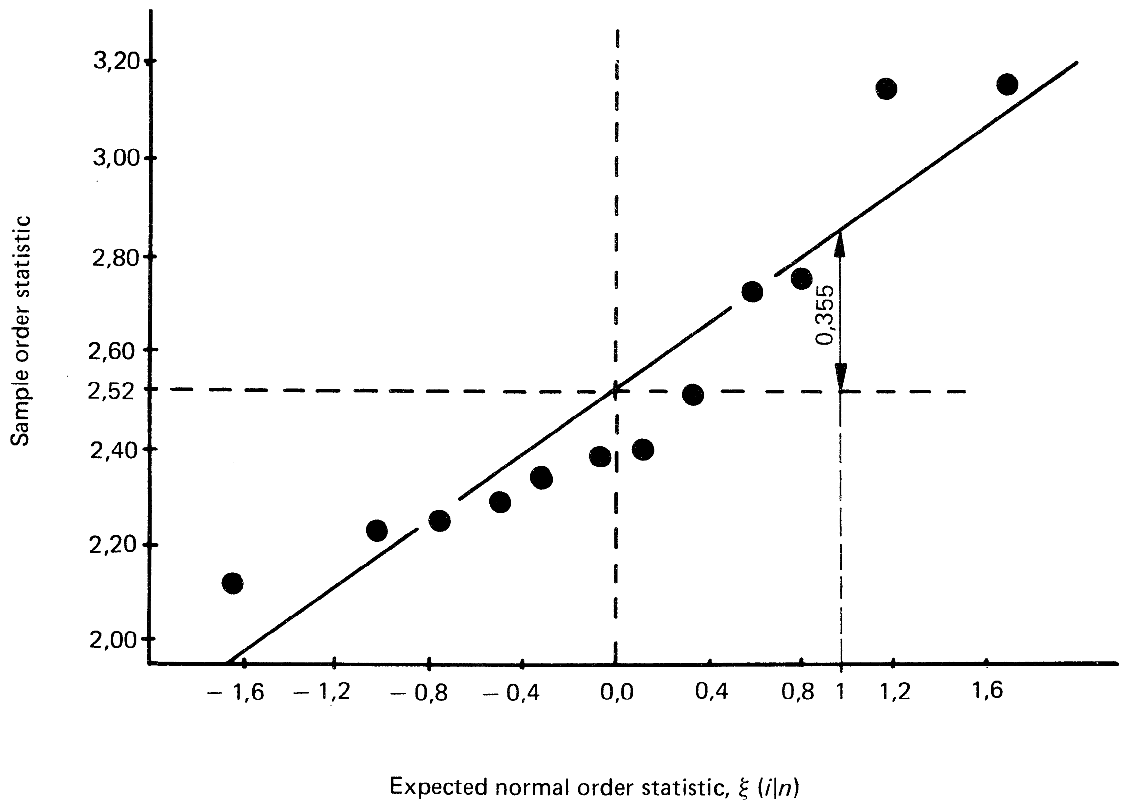


FIGURE 2 – **Graphical test for normality applied to sample of yarn 2**

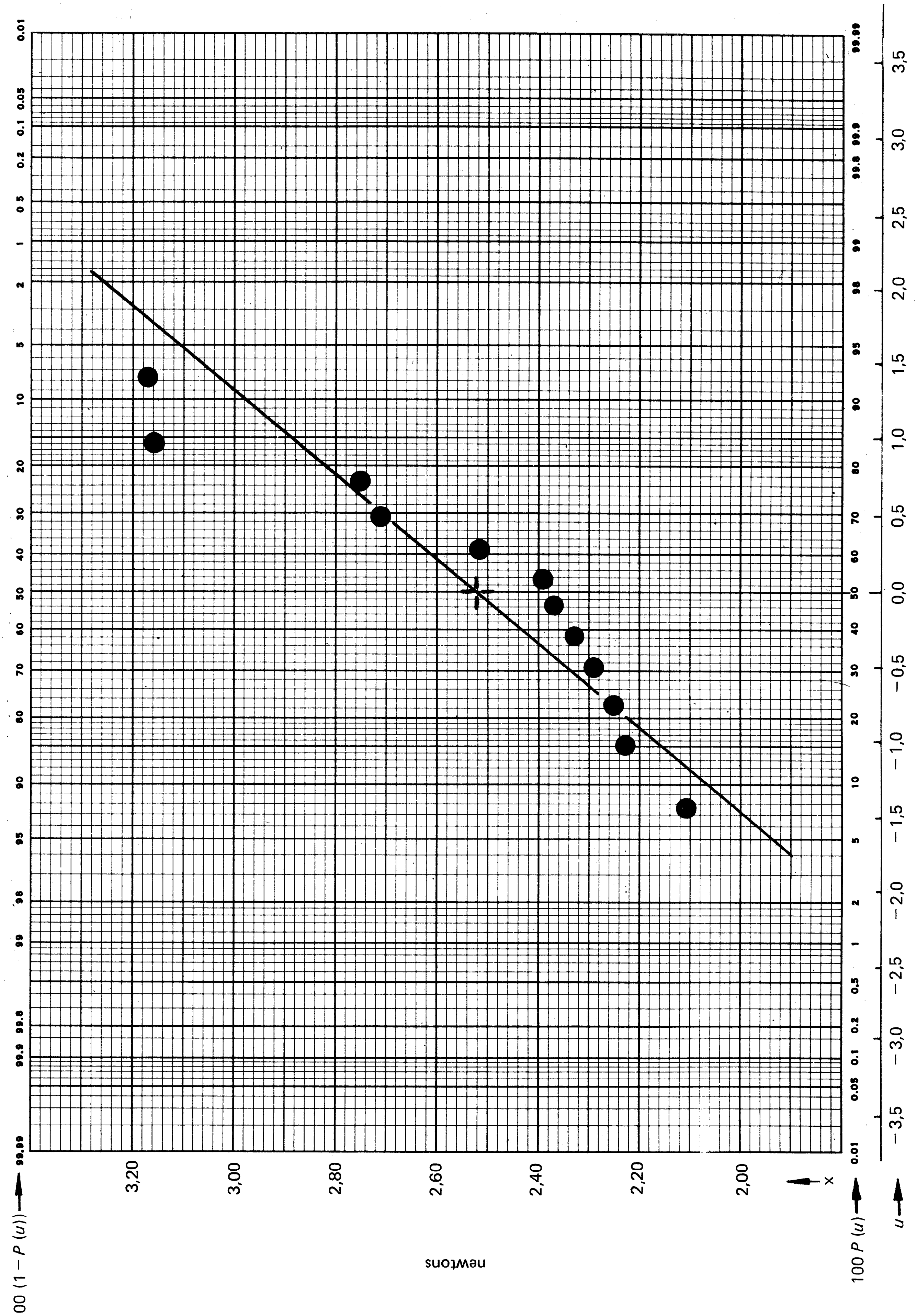


FIGURE 3 — **Illustrating the use of normal probability paper, using plots of 12 values of (x_i, i/13) for data of yarn 2**

In those conditions, if the distribution is strictly normal, the quantiles x_i with probability $i/(n+1)$ of this distribution will be represented graphically by points lying on the straight line passing through the point with co-ordinates (0, m) and having a slope σ. As a consequence, for a single sample, the points with co-ordinates $[i/(n+1), x_i]$ will only diverge through chance fluctuations from this line.

On the other hand, it is clear that in this second method of graphical representation, the mean position of the point representing x_i will not lie on this straight line, although it will be very near to it[1)].

In figure 3, the $n = 12$ ordered observations for the second of the two samples of yarn have been plotted, using a suitable x-scale, against abscissa $P = 1/13, 2/13, \ldots, 12/13$. It will be seen that the spot pattern in figure 3 is very closely similar to that in figure 2, but not precisely so, since $\xi(i|12)$ does not equal $u(P_i = i/13)$ exactly.

The sloping straight line has been drawn using for the unknown population m and σ, the sample estimates $\bar{x} = 2{,}520$, $s = 0{,}355$.

Both these graphical methods may be used if the hypothetical population is not normal but has some other form, for example that of a negative exponential, or a gamma (or χ^2) distribution. But it will then be necessary to have

a) another, appropriate table of the expected values of order statistics, $\xi(i|n)$; or

b) probability paper with a vertical grid drawn to another scale.

Such tables and paper exist.

An alternative graphical method sometimes employed combines elements of the two methods described under a) and b) above. Normal probability paper is again used, the order statistic of the sample, x_i, being plotted as ordinates against abscissa

$$P\{\xi(i|n)\} = \int_{-\infty}^{\xi(i|n)} e^{-u^2/2}\, du/\sqrt{2\pi}$$

instead of against $P_i = i/(n+1)$ as in method b). The values of $P\{\xi(i|n)\}$ may be found by entering a table of the normal probability function with the values of $\xi(i|n)$ given in table V. Again, if the population sampled is normal, the plotted points will lie roughly on a sloping straight line.

The weakness of the graphical method is that it provides no objective means of judging whether, as in this case, the departure of the points from a straight line is important. As stated in paragraph 4 of the "General remarks" introducing section one of this International Standard it is possible to apply the test of Shapiro and Wilk (provided $n \leqslant 50$), which was developed with the idea of giving precision to this graphical approach. This method will be described with others in more detail in a further document. If this test is applied to the observations on yarn 2 and also to the $n = 10$ observations on yarn 1 it is found that in neither case are the results inconsistent with sampling from normal populations.

4) The graphical method described may be particularly helpful in reaching a decision as to whether one of the transformations suggested in paragraph 4 of the "General remarks" is likely to make a variable x more closely normal. As an example of this kind the following data are quoted for the results of a rotating bend fatigue test applied to 15 specimens of an aero-engine component.

The variable, x, measures endurance. If the 15 values of

a) x,

b) $\log_{10}(10\,x)$.

already arranged in ascending order of magnitude, are plotted against the corresponding expected normal order statistics $\xi(i|15)$, $i = 1, 2, \ldots, 15$ taken from table V of annex B, it is at once found (see figure 4) that the plot using $\log x$ is approximately linear, while that for x is decidedly not so. This suggests that in testing hypotheses, the analysis of the kind suggested in tables A, A′, C, C′, E and G should be applied to $\log x$ rather than x. This suggestion was confirmed by fuller test data. If, however, the requirement was to obtain confidence intervals, say, for the mean and standard deviation of x, these could not be derived directly from the confidence intervals for the mean and standard deviation of $\log x$. However, tolerance limits for the whole population of x could be found using $\log x$ as the variate.

Rotating-bend fatigue tests, x and $\log_{10}(10\,x)$

Specimen i	x_i	$\log_{10}(10\,x_i)$
1	0,200	0,301
2	0,330	0,519
3	0,450	0,653
4	0,490	0,690
5	0,780	0,892
6	0,920	0,964
7	0,950	0,978
8	0,970	0,987
9	1,040	1,017
10	1,710	1,233
11	2,220	1,346
12	2,275	1,357
13	3,650	1,562
14	7,000	1,845
15	8,800	1,944

1) The amount by which the true line of means differs from the straight line is greatest when $i = 1$ or n, but is even then small compared with the sample variations about the means, $\xi(i|n)$.

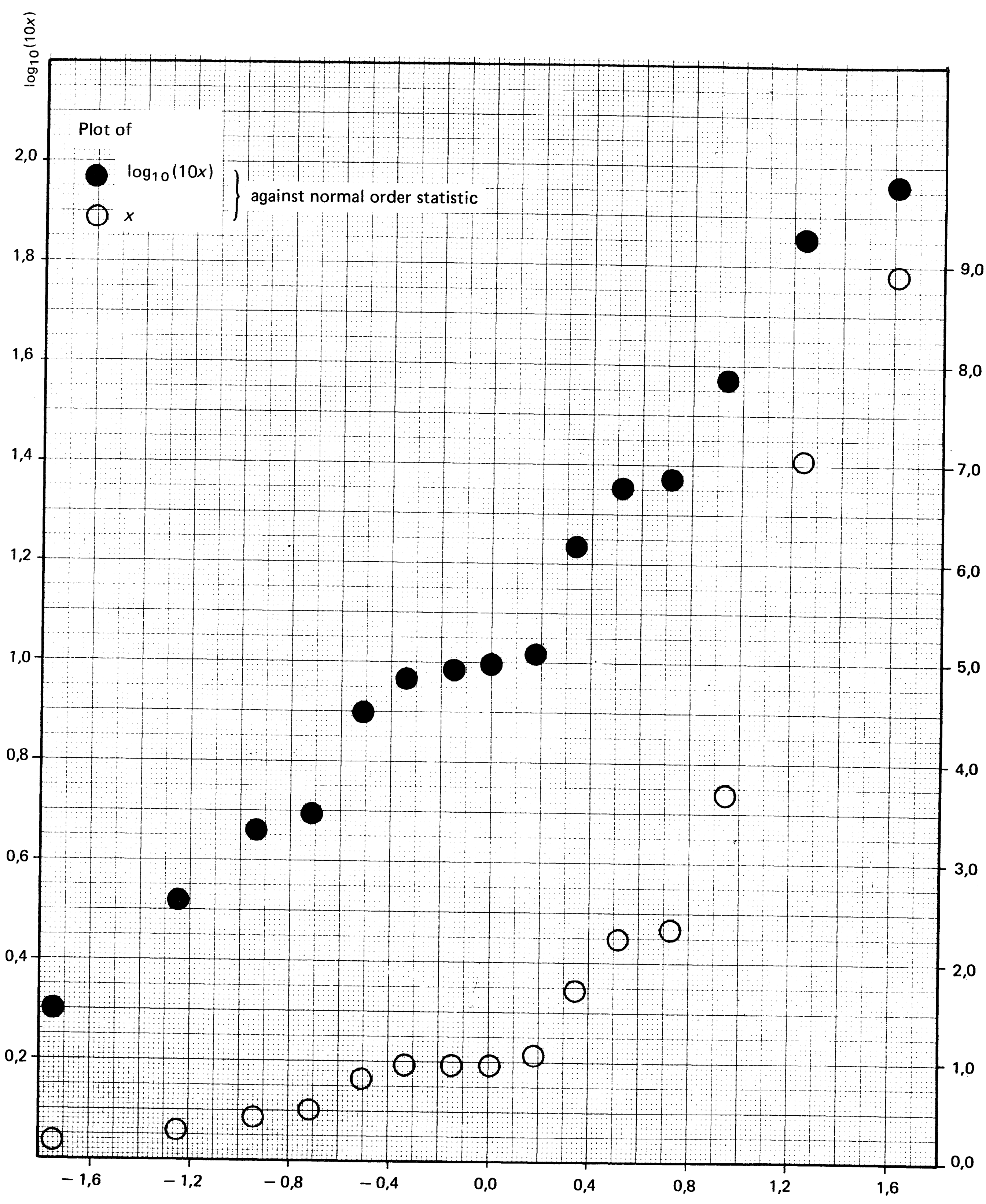


FIGURE 4 – **Rotating-bend fatigue data. Graphical test for normality**

5) Being satisfied, therefore, that it is appropriate to use the procedures described below for the analysis of normally distributed variables, the only pieces of numerical information required from the samples are the number of observations, n (the sample size), and the statistics $\Sigma(x)$ and $\Sigma(x-\bar{x})^2$. These, with their derived sample estimates, the means $m_1{}^* = \bar{x}_1$, $m_2{}^* = \bar{x}_2$, and the variances $\sigma_1^{2*} = s_1^2$ and $\sigma_2^{2*} = s_2^2$, are set out beside the basic data in table X. As previously stated in illustrating the procedure contained in the twelve tables A to H, a complete formal presentation of data and computational workings will only be given for table A (single-sample test on the mean with variance known) and table C (comparison of two means, variances assumed known and not necessarily equal). In the other ten cases the illustration in the following notes will be confined to

a) stating the question to be put to the data;

b) inserting into the formulae of the formulae table the appropriate numerical values taken from table X and from tables I to IV of annex B;

c) discussing the conclusion reached.

6) The methods described above in tables C and C′ concern the comparison of means derived from two completely independent samples. In certain situations, however, the observations in the two samples are related in pairs, say x_i and y_i ($i = 1, 2, \ldots, n$). The problem of practical interest may then be to study the differences $d_i = y_i - x_i$, either in regard to the mean value or the variance of d_i. Problems of this kind will be considered fully in a further document. However, to avoid possible misuse of table C′ where table A′ should be used, an illustration of such a problem is set out in annex A although no formal presentation of the procedure has been given in section one of this International Standard.

7) Finally, it is possible to summarize the relationship between the situations presented in the twelve tables A to H and I to IV of annex B as follows :

a) If the question asked concerns the relationship between sample and/or population means, and the variances are specified or believed known from past experience (tables A, B, C, D), then the procedures can be based on the use of the standardized normal deviate U of table I of annex B;

b) If on the other hand, when dealing with mean values, the variances must be estimated from the sample data (tables A′, B′, C′, D′) then the procedure must be based on the use of the distribution of "Student's" t of table II of annex B. Inevitably in this case, conclusions are reached with somewhat less precision, but it is better that this should be so than that an erroneous value of the variance or standard deviation should be introduced under a) above.

c) If the question asked concerns the relationship between a sample variance and a population variance (tables E, F), then the procedures make use of the distribution of χ^2 of table III of annex B;

d) If it is desired to compare two variances or to derive an estimate of the limits within which the ratio of the two unknown population variances lies (tables G, H), then the procedure makes use of the distribution of the variance ratio F (sometimes called Snedecor's ratio) of table IV of annex B.

NUMERICAL ILLUSTRATION OF PROCEDURES

TABLE A — **Comparison of a mean with a given value** (variance known)

Suppose it is necessary to examine whether the tests on the sample of 10 pieces of yarn (yarn 1 of table X) are consistent with the manufacturer's claim that the mean breaking load of his yarn has a given value, $m_0 = 2{,}40$. It will be supposed that earlier measurements have shown that the variation from consignment to consignment, if not the mean value, is stable and may be represented by a standard deviation of $\sigma = 0{,}331\ 5$. Following the scheme given in table A, the formal presentation of the numerical data would then be as follows :

Technical characteristics of the population studied : The batch consists of a consignment of cotton yarn received on 1969-08-03 from supplier H consisting of 10 000 bobbins packed in 100 boxes with 100 bobbins in each.

Technical characteristics of the sample elements : 10 boxes were drawn at random and one bobbin drawn at random out of each of these boxes. Test pieces of 50 cm length of yarn were cut out from the bobbins at about 5 m distance from the free end. The actual tests were carried out on the central 25 cm of these test pieces, the breaking load in newtons being measured on each piece.

Discarded observations : none

Statistical data

Sample size :

$n = 10$

Sum of the observed values :

$\Sigma x = 21{,}761$

Given value :

$m_0 = 2{,}40$

Known value of the standard deviation :

$\sigma = 0{,}331\,5$

Significance level chosen :

$\alpha = 0{,}05$

Calculations

$$\bar{x} = \frac{21{,}761}{10} = 2{,}176$$

Using table I of annex B,

$$(u_{0,975}/\sqrt{10})\ \sigma = 0{,}620 \times 0{,}331\,5 = 0{,}205\,5$$

Results

Comparison of the population mean with the given value m_0 :

Two-sided case :

$$|\bar{x} - m_0| = |2{,}176 - 2{,}40| = 0{,}224 > 0{,}205\,5$$

The hypothesis that the population mean equals 2,40 is rejected at the 5 % level.

TABLE A′ — **Comparison of a mean with a given value** (variance unknown)

The problem is the same as that described under table A, but in this case the variance must be estimated from the sample, either because no earlier measurements are available or because it is thought that they are no longer appropriate. We apply the formal procedure of table A′ to the data of yarn 1, using the numerical values already given in table X.

In this case $\sigma^* = s = \sqrt{0,139\ 60} = 0,373\ 6$ and
$\sigma^*/\sqrt{10} = 0,118\ 1$, $\nu = 10 - 1 = 9$.

Taking a two-sided test with $\alpha = 0,05$, we find from table IIa of annex B that $t_{0,975}(9) = 2,262$, so that $t_{0,975}(\sigma^*/\sqrt{10}) = 0,267$.

Comparing the sample mean, $\bar{x} = 2,176$ with the manufacturer's claimed value of 2,40, we find

$$|\bar{x} - m_0| = 0,224 < 0,267$$

It follows that the sample results are not inconsistent with the manufacturer's claim. Note that the sample estimate of σ, i.e. $\sigma^* = s = 0,373\ 6$, is larger than that assumed in the illustration of table A ($\sigma = 0,331\ 5$) and as a result we cannot now be confident that the population mean has fallen below 2,40.

If it is preferred to use table IIb, of annex B, giving values of the ratio $t_{1-\alpha/2}(\nu)/\sqrt{n}$ for $\nu = n - 1 = 9$, we must compare $|\bar{x} - m_0|$ with
$[t_{0,975}(9)/\sqrt{10}]^* = 0,715 \times 0,373\ 6 = 0,267$,
the same critical figure as obtained using table IIa.

TABLE B — **Interval estimation of a mean** (variance known)

In this case we do not test whether the population mean has a specified value m_0, but seek limits within which the unknown true mean, m, lies. We then associate a probability $1 - \alpha$ with the statement that the limits include m.

The formal procedure of table B can be applied to the data of yarn 1. It will be supposed that it is again justifiable to use the population standard deviation, derived from earlier measurements, i.e. that $\sigma = 0,331\ 5$. For a two-sided confidence interval associated with a probability $1 - \alpha = 0,95$, we have

$$\bar{x} = 2,176$$

and $(u_{0,975}/\sqrt{10})\sigma = 0,620 \times 0,331\ 5 = 0,205\ 5$

from table I of annex B. It follows that the 95 % confidence interval for m is

$$2,176 - 0,205 < m < 2,176 + 0,205\ 5$$

or $$1,970 < m < 2,382$$

TABLE B′ — **Interval estimation of a mean** (variance unknown)

The problem is the same as that just described except that the estimate $\sigma^* = s$ is substituted for σ and the probability limits of t(or $t/\sqrt{n}$) are used instead of those for $u/\sqrt{n}$.

Applying the procedure of table B′ to derive two-sided confidence limits for m, with $1 - \alpha = 0,95$, using the same sample of yarn 1, we have $n = 10$, $\nu = 9$, $\bar{x} = 2,176$, $s = 0,373\ 6$, $t_{0,975}(s/\sqrt{10}) = 0,267$ as in the illustration of table A′, so that the 95 % confidence interval derived from the sample is given by the statement :

$$2,176 - 0,267 < m < 2,176 + 0,267$$

or $$1,909 < m < 2,443$$

If it is desired to obtain limits, necessarily wider, to which greater confidence can be assigned, we could take $1 - \alpha = 0,99$.

Then table IIa of annex B gives $t_{0,995}(9) = 3,250$ or, alternatively, table IIb of annex B gives $t_{0,995}(9)/\sqrt{10} = 1,028$.

As a result, by either means we find

$$t_{0,995}(s/\sqrt{10}) = (t_{0,995}/\sqrt{10})\,s = 0,384$$

The 99 % confidence interval is now given by the statement

$$2,176 - 0,384 = 1,792 < m < 2,560 = 2,176 + 0,384$$

This interval is clearly wider than that just derived using the scheme of table B, under which it was supposed that the variance was known. This is the penalty which must be paid for having to estimate the variance from a small sample. On the other hand it may be safer to use an estimate derived from the sample if there is any doubt whether the variance based on past experience is still relevant.

TABLE C — **Comparison of two means** (variances known)

This will be illustrated by comparing the means of the samples of yarn 1 and yarn 2 given in table X. It is supposed that the population variances have been satisfactorily established from earlier measurements as

$$\sigma_1^2 = 0,109\ 89,\ \sigma_1 = 0,331\ 5$$

$$\sigma_2^2 = 0,096\ 85,\ \sigma_2 = 0,311\ 2$$

The formal presentation of the numerical data would then be as follows :

Technical characteristics of the population : 2 batches of yarn received on 1969-08-03 from supplier H and on 1969-08-05 from supplier F, consisting of 10 000 and 12 000 bobbins respectively, packed in boxes of 100 bobbins.

Technical characteristics of the samples : 10 and 12 boxes, respectively, were drawn at random from each batch and one bobbin was drawn at random from each of these boxes. Test pieces of 50 cm length were cut at about 5 m distance from the free end of the bobbins sampled. The actual tests were carried out on the central 25 cm of these test pieces, the breaking load in newtons being measured on each piece.

Discarded observations : none.

Statistical data

	First sample	Second sample
Size :		
$n =$	10	12
Sum of the observed values :		
$\Sigma x =$	21,761	30,241
Known value of the variance :		
$\sigma^2 =$	0,109 89	0,096 85
Significance level chosen :		
$\alpha = 0{,}05$		

Calculations

$$\bar{x}_1 = \frac{21{,}761}{10} = 2{,}176$$

$$\bar{x}_2 = \frac{30{,}241}{12} = 2{,}520$$

$$\sigma_d = \sqrt{\frac{0{,}109\,89}{10} + \frac{0{,}096\,85}{12}} = 0{,}138\,1$$

$$u_{0,975}\,\sigma_d = 1{,}96 \times 0{,}138\,1 = 0{,}271$$

Results

Comparison of the two population means :

Two-sided case :

$$|2{,}176 - 2{,}520| = 0{,}344 > 0{,}271$$

The null hypothesis that the means are equal is rejected at the 5 % level. The second type of yarn has the breaking load accepted as the largest.

If we are not prepared to take so large a risk as 0,05, or 1 in 20, of being wrong in our conclusion, we may take $\alpha = 0{,}01$. We then have

$$u_{0,995}\sigma = 2{,}576 \times 0{,}138\,1 = 0{,}356$$

Hence, for the two-sided case

$$|2{,}176 - 2{,}520| = 0{,}344 < 0{,}356$$

and we should not be able to reject the null hypothesis at the 1 % level.

TABLE C' — **Comparison of two means** (variances unknown but may be assumed equal)

The problem differs from that last described because as will commonly happen it is not considered justifiable to accept the values σ_1^2 and σ_2^2 based on previous measurements. It is therefore necessary to obtain an estimate of variance from the sample data. The test is strictly valid only if the two unknown population variances are equal, but it will be very little in error, particularly if the sample sizes n_1 and n_2 are nearly equal, if we use the pooled estimate $\sigma_d{}^*$ quoted in table C'.

The two samples of yarn given in table X will be used and the table need not be repeated. In this case we have

$$\bar{x}_1 = 2{,}176 \qquad \bar{x}_2 = 2{,}520$$

Sums of squares of differences about mean :

		degrees of freedom
1st sample	1,256 365	$10 - 1 = 9$
2nd sample	1,389 769	$12 - 1 = 11$
Total	2,646 134	$22 - 2 = 20$

Estimate of the standard deviation of the difference between $\bar{x}_1$ and $\bar{x}_2$:

$$\sigma_d^* = \sqrt{\frac{22}{10 \times 12} \times \frac{2{,}646\,134}{20}} = 0{,}155\,7$$

Using a two-sided test with $\alpha = 0{,}05$, we have

$$t_{0,975}(20)\ \sigma_d^* = 2{,}086 \times 0{,}155\,7 = 0{,}325$$

$$|\bar{x}_1 - \bar{x}_2| = |2{,}176 - 2{,}520| = 0{,}344 > 0{,}325$$

The hypothesis of equal population means : $m_1 = m_2$ is therefore just rejected at the 5 % level. It would not be rejected at the 1 % level.

TABLE D — **Estimation of the difference of two means** (variances known)

In this case we do not test whether the two populations have a common mean value but use the two samples to estimate the difference between their two means, m_1 and m_2. We obtain confidence limits for this difference, $m_1 - m_2$, associated with a probability $1 - \alpha$.

The same data will again be used and it will be supposed that the variances $\sigma_1^2 = 0{,}109\,89$ and $\sigma_2^2 = 0{,}096\,85$ are known from previous measurements. The standard deviation of the difference in sample means, $\bar{x}_1$ and $\bar{x}_2$, will again as in table C be $\sigma_d = 0{,}138\,1$, and

$$u_{0,975}\sigma_d = 1{,}96 \times 0{,}138\,1 = 0{,}271$$

$$\bar{x}_1 - \bar{x}_2 = -0{,}344$$

It follows that the two-sided 95 % confidence interval for $m_1 - m_2$ is

$$-0{,}344 - 0{,}271 < m_1 - m_2 < -0{,}344 + 0{,}271$$

or

$$0{,}073 < m_2 - m_1 < 0{,}615$$

TABLE D' — **Estimation of the difference of two means** (variances unknown, but may be assumed equal)

It is required to estimate a confidence interval for the difference between the mean breaking loads of the two types of yarn. In this case there are no acceptable values of σ_1^2 and σ_2^2 based on past measurements, but assuming that the unknown variances are equal, or nearly so, we use the common value obtained from the pooled data, as derived above, namely

$$\sigma_d^* = 0{,}155\,7$$

and proceed to find a two-sided confidence interval for $m_1 - m_2$. The procedure is as in table D except that $\sigma_d{}^*$ is substituted for σ_d and $t_{0,975}(\nu)$ for $u_{0,975}$, giving

$$t_{0,975}(20)\ \sigma_d^* = 2{,}086 \times 0{,}155\,7 = 0{,}325$$

This gives the inequality

$$-0{,}344 - 0{,}325 < m_1 - m_2 < -0{,}344 + 0{,}325$$

or

$$0{,}019 < m_2 - m_1 < 0{,}669$$

associated with a probability of $1 - \alpha = 0{,}95$. Note that in this case where the variance has to be estimated, the interval based on the t-distribution is somewhat wider than that found in the example illustrating table D.

TABLE E — **Comparison of a variance with a given value**

The preceding examples have been concerned with relationships between sample and population mean values. In the present example and in the three which follow it is the relationship between sample and population variances or standard deviations which are of interest. Take the 10 observations of breaking load of yarn 1 from table X and ask whether these are consistent with the hypothesis that the population variance does not exceed a specified value of $\sigma_0^2 = 0{,}090\,0$.

This is the one-sided case a) of table E. Using the results given in table X, we have

$$\frac{\Sigma(x - \bar{x})^2}{\sigma_0^2} = \frac{1{,}256\,365}{0{,}090\,0} = 13{,}96$$

Reference to table III of annex B shows that for degrees of freedom $\nu = 9$, the upper 5 % point of χ^2 is 16,92, so that the observed sample variance is not inconsistent with the null hypothesis (that $\sigma^2 \leqslant 0{,}090\,0$). Though the sample variance, $s_1^2 = 0{,}139\,6$ is a good deal larger than the specified value of 0,090 0, such a difference might well occur through chance in a sample of only 10 observations.

TABLE F — **Estimation of a variance**

The data of the sample of yarn 1 may also be used to derive lower and upper confidence limits for the unknown σ^2. If we take $1 - \alpha = 0{,}95$, table III of annex B gives for degrees of freedom $\nu = 9$.

$$\chi^2_{0{,}025}(9) = 2{,}700$$

$$\chi^2_{0{,}975}(9) = 19{,}02$$

Hence

$$\frac{\Sigma(x-\bar{x})^2}{\chi^2_{0{,}025}} = \frac{1{,}256\ 3}{2{,}700} = 0{,}465\ 3$$

$$\frac{\Sigma(x-\bar{x})^2}{\chi^2_{0{,}975}} = \frac{1{,}256\ 3}{19{,}02} = 0{,}066\ 1$$

and a probability of 0,95, or odds of 19 over 20, can be associated with the statement

$$0{,}066\ 1 < \sigma^2 < 0{,}465\ 3, \text{ or } 0{,}257 < \sigma < 0{,}682$$

If it were desired to obtain limits, necessarily wider, for which the probability of including the unknown variance were greater, for example 0,99 instead of 0,95, values of $\chi^2_{0{,}005}(9)$ and $\chi^2_{0{,}995}(9)$ could be obtained from table III of annex B. The confidence limits now become

$$0{,}053\ 26 < \sigma^2 < 0{,}724, \text{or } 0{,}231 < \sigma < 0{,}851$$

TABLE G — **Comparison of two variances**

It is required to determine whether the results for the samples of yarn 1 and yarn 2 given in table X are consistent with the hypothesis that the two populations have a common but unspecified breaking-load variance, $\sigma_1^2 = \sigma_2^2$.

TABLE X gives

$$\nu_1 = 10 - 1 = 9, \nu_2 = 12 - 1 = 11$$

$$s_1^2 = 0{,}139\ 60, s_2^2 = 0{,}126\ 34$$

It follows that $F = s_1^2/s_2^2 = 1{,}10$

From table IV of annex B we find by rough interpolation that

$$F_{1-\alpha/2}(\nu_1, \nu_2) = F_{0{,}975}(9, 11) = 3{,}6$$

$$F_{\alpha/2}(\nu_1, \nu_2) = 1/F_{0{,}975}(11, 9) = 1/4{,}0 = 0{,}25$$

The observed ratio of 1,10 lies well within these limits so that there is no reason to doubt the hypothesis that $\sigma_1^2 = \sigma_2^2$.

TABLE H — **Estimation of the ratio of two variances**

Taking the two samples of breaking load in yarn (data in table X) limits are required for the ratio of population variances, σ_1^2/σ_2^2.

Besides the approximate values

$$F_{0{,}975}(9, 11) = 3{,}6$$

$$F_{0{,}025}(9, 11) = 0{,}25$$

already obtained by interpolation in table IV of annex B in the preceding example, we can similarly find

$$F_{0{,}995}(9, 11) = 5{,}6$$

$$F_{0{,}005}(9, 11) = \frac{1}{F_{0{,}995}(11, 9)} = \frac{1}{6{,}4} = 0{,}16$$

The rule of table H therefore provides the following confidence intervals, since $s_1^2/s_2^2 = 1{,}10$

Confidence level		Limits for ratio of population variances σ_1^2/σ_2^2
0,95		$\frac{1}{3{,}6} \times 1{,}10 = 0{,}31 < \sigma_1^2/\sigma_2^2 < 4{,}4 = 4 \times 1{,}10$
	or	$0{,}56 < \sigma_1/\sigma_2 < 2{,}1$
0,99		$\frac{1}{5{,}6} \times 1{,}10 = 0{,}20 < \sigma_1^2/\sigma_2^2 < 7{,}0 = 6{,}4 \times 1{,}10$
	or	$0{,}45 < \sigma_1/\sigma_2 < 2{,}6$

Again it will be noted that for samples as small as 10 and 12 the limits associated with a confidence level of 0,95 or odds of 19 over 20 are very wide. If greater assurance still is needed (odds of 99 over 100) that the limits will include the unknown true ratio, the limits for the ratio of variances are so wide as to be almost valueless, although expressed as a ratio of standard deviations they do not appear so extreme. In other words, much larger samples are needed to estimate a ratio of variances, or indeed a single variance, with any degree of accuracy.

ANNEX A

COMPARISON OF PAIRED OBSERVATIONS USING STUDENT'S t-TEST

In connection with the procedure illustrated under the headings of tables C′ and D′, it is of importance to note that a different procedure has to be used when the two sets of values, say x_i and y_i, are not independent, but paired. This for example is the case if a single sample of n items is drawn from a population and two observations of the same character are made on each sample element, i, an observation x_i and an observation y_i ($i = 1, 2, \ldots, n$). Usually the latter observation is made after some treatment has been applied and the former before or without the application of treatment. Detecting a difference between the means of the two variates then amounts to assessing an effect of the treatment (or difference in treatments) on the character studied.

The data tabled below were collected in an investigation designed to determine whether the average rate of shaft-wear caused by various bearing metals in an internal combustion engine differed between metals.

(Data from W.E. Duckworth, *Statistical Techniques in Technological Research,* published by Methuen and Co.)

Shaft-wear after a given working time in 0.000 1 in

Trial i)	Wear with white metal (x_i)	Wear with copper lead (y_i)	$d_i = y_i - x_i$
1	1.5	3.5	2.0
2	1.3	2.0	0.7
3	4.5	4.7	0.2
4	2.5	2.8	0.3
5	4.5	6.5	2.0
6	1.7	2.2	0.5
7	1.8	2.5	0.7
8	3.3	5.8	2.5
9	2.3	4.2	1.9
Totals	23.4	34.2	10.8

If these data are treated as two completely independent samples of $n = 9$ observations following the procedure of table C′, it is found that

$$\bar{x} = 2{,}60,\ \Sigma(x - \bar{x})^2 = 12{,}16$$

$$\bar{y} = 3{,}80,\ \Sigma(y - \bar{y})^2 = 20{,}84$$

Following the procedure of table C′, we find

$$t = 1{,}2 \Big/ \sqrt{\frac{12{,}16 + 20{,}84}{16} \times \frac{2}{9}} = 1{,}77$$

With $\nu = 16$ degrees of freedom, table IIa of annex B gives for $\alpha/2 = 0{,}025$, $t_{1-\alpha/2} = 2{,}12$, so that the difference in means is not significant at the 5 % level (two-sided test).

However, as is clear from a comparison of corresponding values x_i and y_i in the table, the observations are correlated in pairs. To eliminate possible effects due to differences in rate of wear on different shafts, the experiment was designed so that in each engine a white-metal bearing and a copper-lead bearing were tested together on the same shaft. This means that the (x_i, y_i) form nine pairs, each derived from metals tested under as nearly the same conditions as possible.

It may be assumed that the common contribution to x_i and y_i due to the two metals being tested together on the same, ith, shaft may be represented by an additive term z_i so that

$$x_i = z_i + v_i,\ y_i = z_i + w_i$$

where v_i and w_i are independent normally distributed chance variables, that is to say

$$d_i = y_i - x_i = w_i - v_i$$

will be normally distributed. The hypothesis tested is that the mean shaft-wear is independent of the metal selected, i.e. if the differences d_i, vary only from chance causes about a mean of zero. To examine this hypothesis we apply the single-sample t-test as in table A′. The nine values of d_i are shown in the last column of the table above, and we find

$$\bar{d} = 10{,}8/9 = 1{,}2$$

$$\Sigma(d_i - d)^2 = 6{,}26$$

$$s_d = \sqrt{6{,}26/8} = 0{,}884\ 6$$

$$\text{Hence, } t = \frac{(\bar{d} - 0)\sqrt{9}}{s_d} = 1{,}2 \times 3/0{,}884\ 6 = 4{,}07$$

From table IIa, with $\nu = 9 - 1 = 8$ degrees of freedom, it is seen that $t_{1-\alpha/2} = 3{,}35$ for $\alpha/2 = 0{,}005$ so that the difference between the mean wear rates of the two metals is now shown to be highly significant, the wear rate for the copper lead being clearly the greater.

In the same way a narrower confidence interval for the mean difference between wear rates could be obtained using the paired differences and the procedure of table B′, rather than following that of table D′.

Note that if the additive relations $x_i = z_i + v_i$, $y_i = z_i + w_i$ are true or approximately true, there is no need for the "shaft effects", z_i, to be normally distributed, as z vanishes in taking the differences. Of course, in the case of comparing the two yarns, this pairing would not be possible. Suppose, however, that it had been wished to compare the effect of two different treatments on the same yarn, the breaking loads could have been determined by giving the two treatments in pairs to short lengths of yarn cut off close together. In this way the effect of possible long-term fluctuations in strength along the whole length of the yarn (represented by the term z_i) could be largely eliminated and the test made more sensitive to a real treatment difference.

ANNEX B

STATISTICAL TABLES

TABLE I – **Values of the ratio** $u_{1-\alpha}/\sqrt{n}$

n	Two-sided case		One-sided case	
	$\frac{u_{0,975}}{\sqrt{n}}$	$\frac{u_{0,995}}{\sqrt{n}}$	$\frac{u_{0,95}}{\sqrt{n}}$	$\frac{u_{0,99}}{\sqrt{n}}$
1	1,960	2,576	1,645	2,326
2	1,386	1,821	1,163	1,645
3	1,132	1,487	0,950	1,343
4	0,980	1,288	0,822	1,163
5	0,877	1,152	0,736	1,040
6	0,800	1,052	0,672	0,950
7	0,741	0,974	0,622	0,879
8	0,693	0,911	0,582	0,822
9	0,653	0,859	0,548	0,775
10	0,620	0,815	0,520	0,735
11	0,591	0,777	0,496	0,701
12	0,566	0,744	0,475	0,671
13	0,544	0,714	0,456	0,645
14	0,524	0,688	0,440	0,622
15	0,506	0,665	0,425	0,601
16	0,490	0,644	0,411	0,582
17	0,475	0,625	0,399	0,564
18	0,462	0,607	0,388	0,548
19	0,450	0,591	0,377	0,534
20	0,438	0,576	0,368	0,520
21	0,428	0,562	0,359	0,508
22	0,418	0,549	0,351	0,496
23	0,409	0,537	0,343	0,485
24	0,400	0,526	0,336	0,475
25	0,392	0,515	0,329	0,465
26	0,384	0,505	0,323	0,456
27	0,377	0,496	0,317	0,448
28	0,370	0,487	0,311	0,440
29	0,364	0,478	0,305	0,432
30	0,358	0,470	0,300	0,425
31	0,352	0,463	0,295	0,418
41	0,306	0,402	0,257	0,363
51	0,274	0,361	0,230	0,326
61	0,251	0,330	0,211	0,298
71	0,233	0,306	0,195	0,276
81	0,218	0,286	0,183	0,258
91	0,205	0,270	0,172	0,244
101	0,195	0,256	0,164	0,231
201	0,138	0,182	0,116	0,164
501	0,088	0,115	0,073	0,104
∞	0	0	0	0

TABLE IIa – **Fractiles of Student's distribution**

ν	Two-sided case		One-sided case	
	$t_{0,975}$	$t_{0,995}$	$t_{0,95}$	$t_{0,99}$
1	12,706	63,657	6,314	31,821
2	4,303	9,925	2,920	6,965
3	3,182	5,841	2,353	4,541
4	2,776	4,604	2,132	3,747
5	2,571	4,032	2,015	3,365
6	2,447	3,707	1,943	3,143
7	2,365	3,499	1,895	2,998
8	2,306	3,355	1,860	2,896
9	2,262	3,250	1,833	2,821
10	2,288	3,169	1,812	2,764
11	2,201	3,106	1,796	2,718
12	2,179	3,055	1,782	2,681
13	2,160	3,012	1,771	2,650
14	2,145	2,977	1,761	2,624
15	2,131	2,947	1,753	2,602
16	2,120	2,921	1,746	2,583
17	2,110	2,898	1,740	2,567
18	2,101	2,878	1,734	2,552
19	2,093	2,861	1,729	2,539
20	2,086	2,845	1,725	2,528
21	2,080	2,831	1,721	2,518
22	2,074	2,819	1,717	2,508
23	2,069	2,807	1,714	2,500
24	2,064	2,797	1,711	2,492
25	2,060	2,787	1,708	2,485
26	2,056	2,779	1,706	2,479
27	2,052	2,771	1,703	2,473
28	2,048	2,763	1,701	2,467
29	2,045	2,756	1,699	2,462
30	2,042	2,750	1,697	2,457
40	2,021	2,704	1,684	2,423
60	2,000	2,660	1,671	2,390
120	1,980	2,617	1,658	2,358
∞	1,960	2,576	1,645	2,326

Taken from E.S. Pearson and H.O. Hartley, *Biometrika Tables for Statisticians,* Vol. I (1954).

NOTE – For interpolation when $\nu > 30$, take $z = 120/\nu$ as argument.

Example :

$\nu = 40$ $z = 120/\nu = 3$ $t_{0,975} = 2,021$

$\nu = 60$ $z = 120/\nu = 2$ $t_{0,975} = 2,000$

$\nu = 50$ $z = 120/\nu = 2,4$ $t_{0,975} = 2,021 - \frac{3-2,4}{3-2}(2,021-2)$

$t_{0,975} = 2,008$

TABLE IIb – **Values of the ratio** $t_{1-\alpha}(\nu)/\sqrt{n}$ **for** $\nu = n - 1$

$\nu = n - 1$	Two-sided case		One-sided case	
	$\frac{t_{0,975}}{\sqrt{n}}$	$\frac{t_{0,995}}{\sqrt{n}}$	$\frac{t_{0,95}}{\sqrt{n}}$	$\frac{t_{0,99}}{\sqrt{n}}$
1	8,985	45,013	4,465	22,501
2	2,434	5,730	1,686	4,021
3	1,591	2,920	1,177	2,270
4	1,242	2,059	0,953	1,676
5	1,049	1,646	0,823	1,374
6	0,925	1,401	0,734	1,188
7	0,836	1,237	0,670	1,060
8	0,769	1,118	0,620	0,966
9	0,715	1,028	0,580	0,892
10	0,672	0,956	0,546	0,833
11	0,635	0,897	0,518	0,785
12	0,604	0,847	0,494	0,744
13	0,577	0,805	0,473	0,708
14	0,554	0,769	0,455	0,678
15	0,533	0,737	0,438	0,651
16	0,514	0,708	0,423	0,626
17	0,497	0,683	0,410	0,605
18	0,482	0,660	0,398	0,586
19	0,468	0,640	0,387	0,568
20	0,455	0,621	0,376	0,552
21	0,443	0,604	0,367	0,537
22	0,432	0,588	0,358	0,523
23	0,422	0,573	0,350	0,510
24	0,413	0,559	0,342	0,498
25	0,404	0,547	0,335	0,487
26	0,396	0,535	0,328	0,477
27	0,388	0,524	0,322	0,467
28	0,380	0,513	0,316	0,458
29	0,373	0,503	0,310	0,449
30	0,367	0,494	0,305	0,441
40	0,316	0,422	0,263	0,378
50	0,281	0,375	0,235	0,337
60	0,256	0,341	0,214	0,306
70	0,237	0,314	0,198	0,283
80	0,221	0,293	0,185	0,264
90	0,208	0,276	0,174	0,248
100	0,197	0,261	0,165	0,235
200	0,139	0,183	0,117	0,165
500	0,088	0,116	0,074	0,104
∞	0	0	0	0

TABLE III – **Fractiles of the chi-squared distirbution**

ν	**Two-sided case**				**One-sided case**			
	$\chi^2_{0,025}$	$\chi^2_{0,975}$	$\chi^2_{0,005}$	$\chi^2_{0,995}$	$\chi^2_{0,05}$	$\chi^2_{0,95}$	$\chi^2_{0,01}$	$\chi^2_{0,99}$
1	0,001	5,023	0,000 039 3	7,879	0,004	3,841	0,000 2	6,635
2	0,051	7,378	0,010	10,597	0,103	5,991	0,020	9,210
3	0,216	9,348	0,072	12,838	0,352	7,815	0,115	11,345
4	0,484	11,143	0,207	14,860	0,711	9,488	0,297	13,277
5	0,831	12,833	0,412	16,750	1,145	11,071	0,554	15,086
6	1,237	14,449	0,676	18,548	1,635	12,592	0,872	16,812
7	1,690	16,013	0,989	20,278	2,167	14,067	1,239	18,475
8	2,180	17,535	1,344	21,955	2,733	15,507	1,646	20,090
9	2,700	19,023	1,735	23,589	3,325	16,919	2,088	21,666
10	3,247	20,483	2,156	25,188	3,940	18,307	2,558	23,209
11	3,816	21,920	2,603	26,757	4,575	19,675	3,053	24,725
12	4,404	23,337	3,074	28,300	5,226	21,026	3,571	26,217
13	5,009	24,736	3,565	29,819	5,892	22,362	4,107	27,688
14	5,629	26,119	4,075	31,319	6,571	23,685	4,660	29,141
15	6,262	27,488	4,601	32,801	7,261	24,996	5,229	30,578
16	6,908	28,845	5,142	34,267	7,962	26,296	5,812	32,000
17	7,564	30,191	5,697	35,719	8,672	27,587	6,408	33,409
18	8,231	31,526	6,265	37,156	9,390	28,869	7,015	34,805
19	8,907	32,852	6,844	38,582	10,117	30,144	7,633	36,191
20	9,591	34,170	7,434	39,997	10,851	31,410	8,260	37,566
21	10,283	35,479	8,034	41,401	11,591	32,671	8,897	38,932
22	10,982	36,781	8,643	42,796	12,338	33,924	9,542	40,289
23	11,689	38,076	9,260	44,181	13,091	35,173	10,196	41,638
24	12,401	39,364	9,886	45,559	13,848	36,415	10,856	42,980
25	13,120	40,647	10,520	46,928	14,611	37,653	11,524	44,314
26	13,844	41,923	11,160	48,290	15,379	38,885	12,198	45,642
27	14,573	43,194	11,808	49,645	16,151	40,113	12,879	46,963
28	15,308	44,461	12,461	50,993	16,928	41,337	13,565	48,278
29	16,047	45,722	13,121	52,336	17,708	42,557	14,257	49,588
30	16,791	46,979	13,787	53,672	18,493	43,773	14,954	50,892

Taken from E.S. Pearson and H.O. Hartley, *Biometrika Tables for Statisticians,* Vol. I (1954).

See note to table IIa.

TABLE IV – **Upper percentage points of F**

Values of $F_{1-\alpha}(\nu_1, \nu_2)$, $\alpha = 0{,}05$

ν_2 \ ν_1	4	5	6	7	8	10	12	15	20	24	30	40	60	120
4	6,39	6,26	6,16	6,09	6,04	5,96	5,91	5,86	5,80	5,77	5,75	5,72	5,69	5,66
5	5,19	5,05	4,95	4,88	4,82	4,74	4,68	4,62	4,56	4,53	4,50	4,46	4,43	4,40
6	4,53	4,39	4,28	4,21	4,15	4,06	4,00	3,94	3,87	3,84	3,81	3,77	3,74	3,70
7	4,12	3,97	3,87	3,79	3,73	3,64	3,57	3,51	3,44	3,41	3,38	3,34	3,30	3,27
8	3,84	3,69	3,58	3,50	3,44	3,35	3,28	3,22	3,15	3,12	3,08	3,04	3,01	2,97
10	3,48	3,33	3,22	3,14	3,07	2,98	2,91	2,85	2,77	2,74	2,70	2,66	2,62	2,58
12	3,26	3,11	3,00	2,91	2,85	2,75	2,69	2,62	2,54	2,51	2,47	2,43	2,38	2,34
15	3,06	2,90	2,79	2,71	2,64	2,54	2,48	2,40	2,33	2,29	2,25	2,20	2,46	2,11
20	2,87	2,71	2,60	2,51	2,45	2,35	2,28	2,20	2,12	2,08	2,04	1,99	1,95	1,90
24	2,78	2,62	2,51	2,42	2,36	2,25	2,18	2,11	2,03	1,98	1,94	1,89	1,84	1,79
30	2,69	2,53	2,42	2,33	2,27	2,16	2,09	2,01	1,93	1,89	1,84	1,79	1,74	1,68
40	2,61	2,45	2,34	2,25	2,18	2,08	2,00	1,92	1,84	1,79	1,74	1,69	1,64	1,58
60	2,53	2,37	2,25	2,17	2,10	1,99	1,92	1,84	1,75	1,70	1,65	1,59	1,53	1,47
120	2,45	2,29	2,17	2,09	2,02	1,91	1,83	1,75	1,66	1,61	1,55	1,50	1,43	1,35

Values of $F_{1-\alpha}(\nu_1, \nu_2)$, $\alpha = 0{,}025$

ν_2 \ ν_1	4	5	6	7	8	10	12	15	20	24	30	40	60	120
4	9,60	9,36	9,20	9,07	8,98	8,84	8,75	8,66	8,56	8,51	8,46	8,41	8,36	8,31
5	7,39	7,15	6,98	6,85	6,76	6,62	6,52	6,43	6,33	6,28	6,23	6,18	6,12	6,07
6	6,23	5,99	5,82	5,70	5,60	5,46	5,37	5,27	5,17	5,12	5,07	5,01	4,96	4,90
7	5,52	5,29	5,12	4,99	4,90	4,76	4,67	4,57	4,47	4,42	4,36	4,31	4,25	4,20
8	5,05	4,82	4,65	4,53	4,43	4,30	4,20	4,10	4,00	3,95	3,89	3,84	3,78	3,73
10	4,47	4,24	4,07	3,95	3,85	3,72	3,62	3,52	3,42	3,37	3,31	3,26	3,20	3,14
12	4,12	3,89	3,73	3,61	3,51	3,37	3,28	3,18	3,07	3,02	2,96	2,91	2,85	2,79
15	3,80	3,58	3,41	3,29	3,20	3,06	2,96	2,86	2,76	2,70	2,64	2,59	2,52	2,46
20	3,51	3,29	3,13	3,01	2,91	2,77	2,68	2,57	2,46	2,41	2,35	2,29	2,22	2,16
24	3,38	3,15	2,99	2,87	2,78	2,64	2,54	2,44	2,33	2,27	2,21	2,15	2,08	2,01
30	3,25	3,03	2,87	2,75	2,65	2,51	2,41	2,31	2,20	2,14	2,07	2,01	1,94	1,87
40	3,13	2,90	2,74	2,62	2,53	2,39	2,29	2,18	2,07	2,01	1,94	1,88	1,80	1,72
60	3,01	2,79	2,63	2,51	2,41	2,27	2,17	2,06	1,94	1,88	1,82	1,74	1,67	1,58
120	2,89	2,67	2,52	2,39	2,30	2,16	2,05	1,94	1,82	1,76	1,69	1,61	1,53	1,43

Values of $F_{1-\alpha}(\nu_1, \nu_2)$, $\alpha = 0{,}01$

ν_2 \ ν_1	4	5	6	7	8	10	12	15	20	24	30	40	60	120
4	15,98	15,52	15,21	14,98	15,80	14,55	14,37	14,20	14,02	13,93	13,84	13,75	13,65	13,56
5	11,39	10,97	10,67	10,46	10,29	10,05	9,89	9,72	9,55	9,47	9,38	9,29	9,20	9,11
6	9,15	8,75	8,47	8,26	8,10	7,87	7,72	7,56	7,40	7,31	7,23	7,14	7,06	6,97
7	7,85	7,46	7,19	6,99	6,84	6,62	6,47	6,31	6,16	6,07	5,99	5,91	5,82	5,74
8	7,01	6,63	6,37	6,18	6,03	5,81	5,67	5,52	5,36	5,28	5,20	5,12	5,03	4,95
10	5,99	5,64	5,39	5,20	5,06	4,85	4,71	4,56	4,41	4,33	4,25	4,17	4,08	4,00
12	5,41	5,06	4,82	4,64	4,50	4,30	4,16	4,01	3,86	3,78	3,70	3,62	3,54	3,45
15	4,89	4,56	4,32	4,14	4,00	3,80	3,67	3,52	3,37	3,29	3,21	3,13	3,05	2,96
20	4,43	4,10	3,87	3,70	3,56	3,37	3,23	3,09	2,94	2,86	2,78	2,69	2,61	2,52
24	4,22	3,90	3,67	3,50	3,36	3,17	3,03	2,89	2,74	2,66	2,58	2,49	2,40	2,31
30	4,02	3,70	3,47	3,30	3,17	2,98	2,84	2,70	2,55	2,47	2,39	2,30	2,21	2,11
40	3,83	3,51	3,29	3,12	2,99	2,80	2,66	2,52	2,37	2,29	2,20	2,11	2,02	1,92
60	3,65	3,34	3,12	2,95	2,82	2,63	2,50	2,35	2,20	2,12	2,03	1,94	1,84	1,73
120	3,48	3,17	2,96	2,79	2,66	2,47	2,34	2,19	2,03	1,95	1,86	1,76	1,66	1,53

Values of $F_{1-\alpha}(\nu_1, \nu_2)$, $\alpha = 0{,}005$

ν_2 \ ν_1	4	5	6	7	8	10	12	15	20	24	30	40	60	120
4	23,15	22,46	21,97	21,62	21,35	20,97	20,70	20,44	20,17	20,03	19,89	19,75	19,61	19,47
5	15,56	14,94	14,51	14,20	13,96	13,62	13,38	13,15	12,90	12,78	12,66	12,53	12,40	12,27
6	12,03	11,46	11,07	10,79	10,57	10,25	10,03	9,81	9,59	9,47	9,36	9,24	9,12	9,00
7	10,05	9,52	9,16	8,89	8,68	8,38	8,18	7,97	7,75	7,65	7,53	7,42	7,31	7,19
8	8,81	8,30	7,95	7,69	7,50	7,21	7,01	6,81	6,61	6,50	6,40	6,29	6,18	6,06
10	7,34	6,87	6,54	6,30	6,12	5,85	5,66	5,47	5,27	5,17	5,07	4,97	4,86	4,75
12	6,52	6,07	5,76	5,52	5,35	5,09	4,91	4,72	4,53	4,43	4,33	4,23	4,12	4,01
15	5,80	5,37	5,07	4,85	4,67	4,42	4,25	4,07	3,88	3,79	3,69	3,58	3,48	3,37
20	5,17	4,76	4,47	4,26	4,09	3,85	3,68	3,50	3,32	3,22	3,12	3,02	2,92	2,81
24	4,89	4,49	4,20	3,99	3,83	3,59	3,42	3,25	3,06	2,97	2,87	2,77	2,66	2,55
30	4,62	4,23	3,95	3,74	3,58	3,34	3,18	3,01	2,82	2,73	2,63	2,52	2,42	2,30
40	4,37	3,99	3,71	3,51	3,35	3,12	2,95	2,78	2,60	2,50	2,40	2,30	2,18	2,06
60	4,14	3,76	3,49	3,29	3,13	2,90	2,74	2,57	2,39	2,29	2,19	2,08	1,96	1,83
120	3,92	3,55	3,28	3,09	2,93	2,71	2,54	2,37	2,19	2,09	1,98	1,87	1,75	1,61

Taken from table 18, *Biometrika Tables for Statisticians,* Vol. 1, 1966.

NOTES

1) For the lower 100 α % points, $F_\alpha(\nu_1, \nu_2) = 1/F_{1-\alpha}(\nu_2, \nu_1)$.

2) For interpolation

a) between ν_1, $\nu_2 = 10$ and 20 take $z = 60/\nu$ as argument;

b) beyond ν_1, $\nu_2 = 20$ take $z' = 120/\nu$ as argument.

TABLE V — **Expected values of normal order statistics,** $\xi(i|n)$

i \ n	3[1]	4	5	6	7	8	9	10	11	12	13	14
1	0,846	1,029	1,163	1,267	1,352	1,424	1,485	1,539	1,586	1,629	1,668	1,703
2	0,000	0,297	0,495	0,642	0,757	0,852	0,932	1,001	1,062	1,116	1,164	1,208
3			0,000	0,202	0,353	0,473	0,572	0,656	0,729	0,793	0,850	0,901
4					0,000	0,153	0,275	0,376	0,462	0,537	0,603	0,662
5							0,000	0,123	0,225	0,312	0,388	0,456
6									0,000	0,103	0,191	0,267
7											0,000	0,088

i \ n	15	16	17	18	19	20	21	22	23	24	25	26
1	1,736	1,766	1,794	1,820	1,844	1,867	1,889	1,910	1,929	1,948	1,965	1,982
2	1,248	1,285	1,319	1,350	1,380	1,408	1,434	1,458	1,481	1,503	1,524	1,544
3	0,948	0,990	1,029	1,066	1,099	1,131	1,160	1,188	1,214	1,239	1,263	1,285
4	0,715	0,763	0,807	0,848	0,886	0,921	0,954	0,985	1,014	1,041	1,067	1,091
5	0,516	0,570	0,619	0,665	0,707	0,745	0,781	0,815	0,847	0,877	0,905	0,932
6	0,335	0,396	0,451	0,502	0,548	0,590	0,630	0,667	0,701	0,734	0,764	0,793
7	0,165	0,234	0,295	0,351	0,402	0,448	0,491	0,532	0,569	0,604	0,637	0,668
8	0,000	0,077	0,146	0,208	0,264	0,315	0,362	0,406	0,446	0,484	0,519	0,553
9			0,000	0,069	0,131	0,187	0,238	0,286	0,330	0,370	0,409	0,444
10					0,000	0,062	0,118	0,170	0,218	0,262	0,303	0,341
11							0,000	0,056	0,108	0,156	0,200	0,241
12									0,000	0,052	0,100	0,144
13											0,000	0,048

i \ n	27	28	29	30	31	32	33	34	35	36	37	38
1	1,998	2,014	2,029	2,043	2,056	2,070	2,082	2,095	2,107	2,118	2,129	2,140
2	1,563	1,581	1,599	1,616	1,632	1,647	1,662	1,676	1,690	1,704	1,717	1,729
3	1,306	1,327	1,346	1,365	1,383	1,400	1,416	1,432	1,448	1,462	1,477	1,491
4	1,115	1,137	1,158	1,179	1,198	1,217	1,235	1,252	1,269	1,285	1,300	1,315
5	0,957	0,981	1,004	1,026	1,047	1,067	1,087	1,105	1,123	1,140	1,157	1,173
6	0,820	0,846	0,871	0,894	0,917	0,938	0,959	0,979	0,998	1,016	1,034	1,051
7	0,697	0,725	0,751	0,777	0,801	0,824	0,846	0,867	0,887	0,906	0,925	0,943
8	0,584	0,614	0,642	0,669	0,694	0,719	0,742	0,764	0,786	0,806	0,826	0,845
9	0,478	0,510	0,540	0,568	0,595	0,621	0,646	0,670	0,692	0,714	0,735	0,755
10	0,377	0,411	0,443	0,473	0,502	0,529	0,556	0,580	0,604	0,627	0,649	0,670
11	0,280	0,316	0,350	0,382	0,413	0,442	0,469	0,496	0,521	0,545	0,568	0,590
12	0,185	0,224	0,260	0,294	0,327	0,358	0,387	0,414	0,441	0,466	0,490	0,514
13	0,092	0,134	0,172	0,209	0,243	0,276	0,307	0,336	0,364	0,390	0,416	0,440
14	0,000	0,044	0,086	0,125	0,161	0,196	0,228	0,259	0,289	0,317	0,343	0,369
15			0,000	0,041	0,080	0,117	0,151	0,184	0,215	0,245	0,273	0,300
16					0,000	0,039	0,076	0,110	0,143	0,174	0,203	0,232
17							0,000	0,037	0,071	0,104	0,135	0,165
18									0,000	0,035	0,067	0,099
19											0,000	0,033

i \ n	39	40	41	42	43	44	45	46	47	48	49	50
1	2,151	2,161	2,171	2,180	2,190	2,199	2,208	2,216	2,225	2,233	2,241	2,249
2	1,741	1,753	1,765	1,776	1,787	1,797	1,807	1,817	1,827	1,837	1,846	1,855
3	1,504	1,517	1,530	1,542	1,554	1,565	1,577	1,588	1,598	1,609	1,619	1,629
4	1,330	1,344	1,357	1,370	1,383	1,396	1,408	1,420	1,431	1,442	1,453	1,464
5	1,188	1,203	1,218	1,232	1,246	1,259	1,272	1,284	1,296	1,308	1,320	1,331
6	1,067	1,083	1,099	1,114	1,128	1,142	1,156	1,169	1,182	1,194	1,207	1,218
7	0,960	0,977	0,993	1,009	1,024	1,039	1,054	1,068	1,081	1,094	1,107	1,119
8	0,863	0,881	0,898	0,915	0,931	0,946	0,961	0,976	0,990	1,004	1,017	1,030
9	0,774	0,793	0,811	0,828	0,845	0,861	0,877	0,892	0,907	0,921	0,935	0,949
10	0,690	0,710	0,729	0,747	0,764	0,781	0,798	0,814	0,829	0,844	0,859	0,873
11	0,611	0,632	0,651	0,671	0,689	0,707	0,724	0,740	0,757	0,772	0,787	0,802
12	0,536	0,557	0,578	0,598	0,617	0,636	0,654	0,671	0,688	0,704	0,720	0,735
13	0,463	0,486	0,507	0,528	0,548	0,568	0,586	0,604	0,622	0,639	0,655	0,671
14	0,393	0,417	0,439	0,461	0,482	0,502	0,522	0,540	0,559	0,576	0,593	0,610
15	0,325	0,350	0,373	0,396	0,418	0,439	0,459	0,479	0,498	0,516	0,534	0,551
16	0,258	0,284	0,309	0,333	0,355	0,377	0,398	0,419	0,438	0,457	0,476	0,494
17	0,193	0,220	0,246	0,270	0,294	0,317	0,339	0,360	0,381	0,400	0,419	0,438
18	0,128	0,156	0,183	0,209	0,234	0,258	0,281	0,303	0,324	0,345	0,364	0,384
19	0,064	0,094	0,122	0,149	0,175	0,200	0,224	0,247	0,269	0,290	0,310	0,330
20	0,000	0,031	0,061	0,089	0,116	0,142	0,167	0,191	0,214	0,236	0,257	0,278
21			0,000	0,030	0,058	0,085	0,111	0,136	0,160	0,183	0,205	0,227
22					0,000	0,028	0,055	0,081	0,106	0,130	0,153	0,176
23							0,000	0,027	0,053	0,078	0,102	0,125
24									0,000	0,026	0,051	0,075
25											0,000	0,025

1) For $n = 2$, $\xi(1|2) = 0,564$.

Taken from H.L. Harter, *Order Statistics and their Use in Testing and Estimations, Volume 2.*

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