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General metrology —

Part 4: Practical guide to measurement uncertainty

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Committees responsible for this Published Document

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City University

Environment Agency

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Foreword

This Published Document has been prepared by Technical Committee SS/6. This part of PD 6461 fulfils an identified need to provide clear guidance to scientists, engineers and technicians, who might not be fully conversant with detailed statistical techniques, on how to perform an uncertainty evaluation.

Reports on measurement uncertainty have accompanied test results produced by National Standards Laboratories for many years where highly accurate measurements are required. Furthermore, this practice has recently been extended to results produced by accredited calibration and test facilities.

However, as processes become more efficient, performance specifications become tighter and, with the increasing requirement for traceable measurements, more and more engineers and technicians find they are faced with the task of carrying out an uncertainty evaluation on their reported measurement results.

Guidance on measurement uncertainty exists at a very high level in the form of a document entitled *Guide to the expression of uncertainty in measurement* (GUM) and is available from BSI as PD 6461-3:1995.

This document is intended to provide guidance to practising engineers where no other guidance is available. This document is directed at the practising engineer and others who wish to perform a relatively simple evaluation. It does not replace or supersede any published technical or sector-specific standards on the subject, nor does it replace the GUM, which is regarded as the ultimate authority in the area.

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Summary of pages

This document comprises a front cover, an inside front cover, pages i and ii pages 1 to 41 and a back cover.

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1 Scope

This Published Document sets out the procedures to be followed in evaluating the uncertainty in the result of a measurement. It follows the principles set out in the document *Guide to the expression of uncertainty in measurement*, known as the GUM (PD 6461-3). It is intended for use by the practising measurement professional who seeks a practical guide to the application of the GUM's methodology to industrial measurement problems. This Published Document is applicable to situations where a formula or mathematical model is available for the required quantity in terms of input (or influence) quantities. It does not cover circumstances where reproducibility studies are required to provide a full statement of uncertainty.

Where this Published Document stops short of the rigorous approach advocated in the GUM, reference is made to the appropriate clause in the GUM for those requiring details of the full analysis.

Short examples illustrating the individual steps in the procedures are given in the body of the standard and a fuller example of the application of the procedures is given in Annex A.

2 Normative references

The following referenced documents are indispensable for the application of this document. For dated references, only the cited edition applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

PD 6461-1, *General metrology — Part 1: Basic and general terms* (VIM).

PD 6461-3, *General metrology — Part 3: Guide to the expression of uncertainty in measurement* (GUM).

3 Terms and definitions

For the purposes of this Published Document, the terms and definitions given in PD 6461-1 (VIM) and PD 6461-3 (GUM) and the following apply. References are given to the appropriate clauses of the VIM or the GUM. A small number of additional terms are used in this Published Document and these are defined in **3.2**, **3.4** and **3.11** to **3.13**.

3.1
uncertainty
parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand (VIM **3.9**)

3.2
relative uncertainty
uncertainty expressed as a proportion or percentage of the measurement result

3.3
standard uncertainty
$u(x)$
uncertainty of the result of a measurement expressed as a standard deviation (GUM **2.3.1**)

3.4
relative standard uncertainty
$u^*(x)$
standard uncertainty expressed as a proportion or percentage of the measurement result

3.5
combined standard uncertainty
$u_c(y)$
standard uncertainty of the result of a measurement when that result is obtained from the values of a number of other quantities, equal to the positive square root of a sum of terms, the terms being the variances or covariances of these other quantities weighted according to how the measurement result varies with changes in these quantities (GUM **2.3.4**)

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3.6
expanded uncertainty
***U* = *ku*$_c$(*y*)**
quantity defining an interval about the result of a measurement that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand (GUM **2.3.5**)

NOTE The fraction may be viewed as the coverage probability or the level of confidence of the interval.

3.7
coverage factor
k
numerical factor used as a multiplier of the combined standard uncertainty in order to obtain an expanded uncertainty (GUM **2.3.6**)

NOTE k is typically in the range 2 to 3.

3.8
type A evaluation of uncertainty
method of evaluation of uncertainty by the statistical analysis of a series of observations (GUM **2.3.2**)

3.9
type B evaluation of uncertainty
method of evaluation of uncertainty by means other than the statistical analysis of a series of observations (GUM **2.3.3**)

3.10
sensitivity coefficient
***c*$_i$**
equal to the partial derivative, $\partial f/\partial X_i$, of the functional relationship, f, between the input quantities, $x_1 \ldots x_N$, and the output quantity, y, evaluated at the nominal values of the input quantities (GUM **5.1.3**)

NOTE c_i is used as a weighting factor used to convert uncertainty in an input quantity, x_i, into the resulting uncertainty in the output quantity, y.

3.11
relative sensitivity coefficient
***c**$_i$**
weighting factor used to convert relative uncertainty in an input quantity, x_i, into the resulting relative uncertainty in the output quantity, y

NOTE c^*_i is equal to $c_i x_i / y$.

3.12
contribution to uncertainty
product of the standard uncertainty $u(x_i)$ in input quantity x_i and the absolute value of the sensitivity coefficient c_i

3.13
contribution to relative uncertainty
product of the relative standard uncertainty $u^*(x_i)$ in input quantity x_i and the absolute value of the relative sensitivity coefficient c^*_i

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4 Symbols

a_i	estimated half-width of a component of uncertainty associated with input estimate x_i
b_1, b_2	lower and upper bounds respectively of an asymmetric distribution (**8.3.6**)
c_i	sensitivity coefficient for input quantity x_i
c^*_i	relative sensitivity coefficient for input estimate x_i
f	mathematical relationship between the measurand Y and the input quantities X_i on which Y depends
$\partial f/\partial x_i$	partial derivative with respect to input quantity X_i of the mathematical relationship f between the measurand and the input quantities, given by evaluating $\partial f/\partial X_i$ at $X_1 = x_1, \ldots, X_N = x_N$
k	multiplier or coverage factor used to calculate the expanded uncertainty U
n	number of repeat readings or observations
N	number of input quantities X_i on which the measurand depends
$s(x_j)$	experimental standard deviation of a random variable X_j determined from n repeated observations
$s(\bar{x})$	experimental standard deviation of the arithmetic mean $\bar{x}$
U	expanded uncertainty associated with the output estimate y
U^*	relative expanded uncertainty associated with the output estimate y
$u(x_i)$	standard uncertainty associated with the input estimate x_i
$u_c(y)$	combined standard uncertainty associated with the output estimate y
$u^*(x_i)$	relative standard uncertainty associated with the input estimate x_i
$u_c^*(y)$	combined relative standard uncertainty associated with the output estimate y
$x_{i,j}$	j-th observation of random quantity X_i
$\overline{x_i}$	arithmetic mean or average of n repeated observations $x_{i,j}$, $j = 1,\ldots n$ of input quantity X_i or randomly varying quantity X_i
x_i	estimate of the input quantity X_i
y	estimate of the measurand Y
Δx_i	change in x_i used to calculate the sensitivity coefficient numerically
Δy	change in y determined when calculating the sensitivity coefficient numerically
v	degrees of freedom in the experimental standard deviation of a random variable x determined from n repeated observations. v is equal to $n - 1$

5 The evaluation of uncertainty

The analysis of most measurement uncertainties can be reduced to a common, simple and logical, step-by-step procedure. This procedure is equally applicable to a simple measurement such as the length of a machined component and to an evaluation of the uncertainty in the National Standard for length; the difference between the two analyses lies only in the number, nature and magnitude of the uncertainties to be considered.

The procedure comprises the following steps:

a) definition of the relationship between all the input measurements and the measurand (Clause **6**);

b) for each quantity, compilation of a list of all the factors that contribute to uncertainty in that input (Clause **7**);

c) for each of the input uncertainty sources, estimation of the magnitude of the uncertainty (Clause **8**);

d) from the relationship defined in step a), estimation of the effect that each quantity has on the measurement result (Clause **9**);

e) combining the uncertainties in all the input quantities to obtain the uncertainty in the output quantity (Clause **10**);

f) expressing the expanded uncertainty as an interval about the measurement result within which it is anticipated, with a stated level of confidence, that the measurand will lie (Clause **11**).

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6 Defining the relationship between the input quantities and the output quantity

6.1 It is rare that the required value of a measurand, Y, is obtained directly and in most cases the result is obtained from a measurement of a number, N, of other measurements, X_1, X_2, …, X_N, that are combined through a functional relationship, f, to arrive at the required value. Thus

$$Y = f(X_1, X_2, \ldots, X_N)$$

In practice the input quantities, X_1, X_2, …, X_N, can only be estimated and the result of the measurement is an estimate of Y given by

$$y = f(x_1, x_2, \ldots, x_N)$$

The uncertainties, $u_{x,1}$, $u_{x,2}$, …, $u_{x,N}$, in the input estimates then give rise to the uncertainty in the output estimate, y.

6.2 The functional relationship, or model, can vary enormously in its complexity, from that of the simple traditional balance in which the measurand is compared directly with reference weights to that of a multiphase flow meter measuring a flow of oil, water and gas from an oil well and requiring measurements of pressures, temperatures and densities to derive the flow rates. However complex the relationship, the purpose of setting it down at the start of the uncertainty analysis is to draw up a definitive list of the input quantities. In some cases, the relationship will be defined by the theory of the instrument; in others, it will be defined by the calibration process and the resulting calibration equation; in the most complex cases the calibration process may involve computational techniques such as neural networks and the relationship will then be embedded within the resulting software; the input quantities are then, as with any calibration, the inputs to the calibration process.

6.3 In many cases, it will become apparent that the functional relationship involves quantities that themselves are the result of several measurement inputs. At first sight this might appear to complicate the analysis of the measurement process but, in reality, it is easily handled by reviewing each quantity and defining its relationship to its own inputs. In this way, the most complex measurement process can be sub-divided into manageable stages and the result of each sub-analysis can be carried forward to the next stage. An example of this process is given in Annex A.

NOTE Information on reproducibility studies is available in ISO/TS 21748.

7 Listing the factors affecting each input quantity

7.1 General

Modern industrial instrumentation covers the whole range of scientific measurement, from length, mass and time to electrical capacitance and radiation intensity. The presentation here of a definitive list of sources of uncertainty is therefore not possible. However, it is vital to a rigorous uncertainty analysis that it is recognized that the sources extend far beyond the simple reading of an instrument and indeed in a modern well-calibrated instrumentation system the basic instrument uncertainties might well prove to be relatively small in comparison with some of the other sources of uncertainty. The following subclauses (**7.2** to **7.10**) provide an indication of the sorts of influences that need to be considered in listing the sources of uncertainty in each quantity.

7.2 The measured quantity itself

Some quantities such as mass might well be stable during the measurement process but others can vary considerably; for example, a flowrate might vary as a pump speed varies due to fluctuations in the electrical input to the motor, or a temperature might be affected by changes in some process upstream of the measurement point. In such cases, it is necessary to consider how well the value measured represents the mean of the fluctuating signal. When dealing with periodic signals, it is important to consider the frequency of the variation and ensure that the measurement frequency is significantly greater (typically at least four times as great) to ensure that the measurement is not biased, for example, by repeatedly taking the readings close to the maximum or minimum of the signal. Batch sampling presents special problems and when, for example, drawing a sample of chemical feedstock for a purity analysis, it is important to consider how well the sample taken represents the bulk of the material.

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7.3 The instrument itself

An instrument only provides an estimate of the value of its input. The quality of that estimate depends on many aspects of the instrument; for example, non-linearity, hysteresis, resolution, bias, and drift. Some of these, such as bias, drift and non-linearity, can be reduced by calibration (see **7.4**). Resolution can be subsumed into the spread of values obtained to define a fluctuating signal. Others, like hysteresis, can be difficult to eliminate. Even when an instrument is carefully calibrated, it is important to recognize that the drift process continues and drift between calibrations should be considered.

7.4 Calibration uncertainties

When an instrument is first brought into service, the relationship between the quantity being measured and the instrument output is defined in a calibration experiment. This may range in rigour and complexity from a simple two-point check, zero and one other point, at the end of the production line to a careful definition of a complex calibration curve through many reference points and with many repeats. However, as noted in **7.3**, even when defined by the most rigorous initial calibration, the relationship between input and output will change with time and use and, as a result, uncertainties will increase. These can be reduced by routine periodic calibrations, the purpose of which is to re-establish the relationship between input and output and so replace the unknown uncertainties of a measurement with a quantifiable uncertainty from a defined and controlled calibration experiment. In transferring the uncertainty of the calibration process to the service instrument it should be recognized that, no matter how rigorous the calibration, the uncertainty of the service instrument can never be smaller than the uncertainty of the calibration. It is therefore important that the uncertainty of the calibration is defined, stated on the calibration certificate and included in the uncertainty budget when the instrument is used in normal service.

7.5 Environmental influences

A great many instruments involve electrical measurements and so can be influenced by electromagnetic interference and instabilities in the electrical input. Meters can also be affected by changes in pressure, temperature, humidity and vibration.

7.6 Usage effects

The calibration process is frequently carried out in a test laboratory where the instrument is not necessarily subjected to many of the factors that it will encounter in service. For example, a temperature probe will be calibrated in an indoor environment but, in service, may be employed in a position exposed to the weather and so may be subjected to cooling by wind or heating by solar radiation; or, when being calibrated, a flow meter will be presented with ideal flow conditions, whereas, in service, pipe bends or valves upstream of the meter may introduce asymmetry or swirl into the flow and these may affect its performance. *In situ* calibration may be possible in some situations but, where this is not the case, it is important either to replicate the service conditions in the calibration laboratory or to have a good understanding of the additional effects, so that the effect of the differences between the calibration situation and in-service conditions can be assessed.

7.7 The measurement process

Quality procedures should define the way in which measurements are performed, but there might be circumstances in which measurements have to be taken at a different time or with a different instrument from that defined in the procedures. Such changes in procedure will result in a change in uncertainty.

7.8 Data acquisition and processing

Computer-based data acquisition systems introduce their own sources of uncertainty; analogue-to-digital converters introduce a resolution uncertainty, inter-channel multiplexers have a settling time that can introduce uncertainties into high-speed data logging. Round-off with computer software routines can introduce uncertainty in the processing of data, as can the values of physical constants taken from reference books. When physical property data is taken from published tables or equations, it should be remembered that the uncertainties of the original experiments and of the fitted equations are carried forward into the calculations.

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7.9 Operator effects

Although automated data-logging is now common practice, there are still many situations where instruments are read by a human operator. Faced with a flickering needle or a changing digital display every operator will have his own technique for determining the mean reading. Even the differences in parallax resulting from changes in eye level can introduce variations between the results obtained by different operators.

7.10 Additional sources

The classifications of uncertainty sources listed in **7.2** to **7.9** are intended only as a guide to the possible sources of uncertainty within any measurement. In undertaking an uncertainty analysis, it is important to consider whether there are special sources that pertain to the particular measurements being made.

8 Assessing the magnitude of the uncertainty in each source

8.1 General

The sources of uncertainty listed in **7.2** to **7.10** are many and disparate and there is no one method that can be used to assign numerical values to such a wide variety of effects. The GUM recognizes two basic approaches. The first is based on a statistical analysis of a series of readings: an approach referred to in the GUM as a Type A evaluation of uncertainty (GUM **4.2**). The second uses past experience and professional judgement to arrive at numerical limits: an approach referred to in the GUM as a Type B evaluation of uncertainty (GUM **4.3**). The GUM definitions will be used in the remainder of this Guide when referring to methods of evaluating uncertainties.

8.2 Type A evaluation of uncertainties

8.2.1 *Best estimate of a measurand from a series of readings*

When the value of a measurand is determined by taking a series of measurements, the best estimate of the measurand is taken to be the arithmetic mean of the values recorded. Thus, if the readings of measurand, X_i, are $x_{i,1}$, $x_{i,2}$, $x_{i,3}$, …, $x_{i,j}$, … $x_{i,n}$, the best estimate of X_i is

$$\bar{x}_i = \frac{x_{i,1} + x_{i,2} + \ldots + x_{i,n}}{n} = \frac{1}{n}\sum_{j=1}^{n} x_{i,j} \qquad (1)$$

Example 1: The diameter of a cylinder is determined by measuring the diameter across two mutually perpendicular diameters at each of three planes equally spaced along the axis of the cylinder. The values obtained are

83.17 mm, 83.19 mm, 83.16 mm, 83.17 mm, 83.18 mm, 83.17 mm

The best estimate of the mean diameter is obtained from the values recorded as follows:

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{1}{n}\sum_{j=1}^{n} x_j$$

$$= \frac{1}{6}\ (83.17 + 83.19 + 83.16 + 83.17 + 83.18 + 83.17)$$

$$= 83.173\ \text{mm}$$

Example 2: The temperature within an environmental test chamber is recorded at two-hourly intervals during a 24 h test cycle. The values obtained are

20.9 °C, 19.1 °C, 20.4 °C, 21.2 °C, 20.5 °C, 20.6 °C, 18.6 °C, 19.8 °C, 21.1 °C, 18.8 °C, 19.6 °C, 19.4 °C

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The best estimate of the mean temperature during the 24 hour period is obtained from the values recorded as follows:

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{1}{n}\sum_{j=1}^{n} x_j$$

$$= \frac{1}{12}\ (20.9 + 19.1 + 20.4 + \ldots.. + 19.6 + 19.4)$$

$$= 20.00\ \ ^{o}\mathrm{C}$$

8.2.2 *Standard deviation of a series of measurements*

While the mean of the values recorded can be expected to give a better estimate of the true value than any individual value, the fact that a range of values was obtained raises the question of whether the same mean would have been obtained if more values had been recorded and therefore how well the current mean represents the true value. What is therefore needed is a definition of an interval about the current mean within which the true value might be expected to lie. The range could be defined by the highest and lowest values recorded but this would ignore the majority of the values available and would also be vulnerable to errors, which are likely to result in extreme high or low values. The statistical approach is to calculate the deviation d_i of each value from the mean of the set, $d_i = x_i - \bar{x}$, and from these values to calculate a statistical parameter called the estimated standard deviation, s, of the sample (or set of values).

$$s(x) = \sqrt{\frac{\sum_{i=1}^{n} {d_i}^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}} \qquad (2)$$

The use of $n - 1$ in the denominator corrects for the fact that the value of s is only an estimate of the true value of the standard deviation and is based on a limited sample of readings.

Note that some spreadsheet packages can use a further form of equation (2) that is unsuitable for use when the scatter is small and can lead to significant errors when the experimental scatter is less than one part per million of the mean. The solution is to calculate the individual deviations from the mean and use the second form of equation (2).

Example 3: The standard deviation of the cylinder diameters of example 1 is obtained from the second form of equation (2) as follows

$$s(x) = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}}$$

$$= \sqrt{\frac{(83.17 - 83.173)^2 + (83.19 - 83.173)^2 + \ldots + (83.17 - 83.173)^2}{6-1}}$$

$$= 0.0103\ \ \mathrm{mm}$$

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Example 4: The standard deviation of the measurements of the temperature in the environmental chamber of example 2 is obtained from the second form of equation (2) as follows

$$s(x) = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

$$= \sqrt{\frac{(20.9-20.0)^2 + (19.1-20.0)^2 + \ldots + (19.4-20.0)^2}{12-1}}$$

$$= 0.905 \ {}^{o}\mathrm{C}$$

In most industrial measurement situations s will be obtained from a special function on a calculator or from a function within a spreadsheet program. Care is needed when applying these functions as the alternative form using n as the denominator is often also available to give the standard deviation assuming that the set of data form the entire population – a parameter required for entirely different statistical purposes. If it is unclear which function is available, a useful check is to calculate the standard deviation of the numbers 1 and 2: if the answer given is 0.707, the function is the correct one, but the answer 0.5 indicates that the n denominator is being used and all calculations of the standard deviation will need to be corrected by multiplying by $\sqrt{\frac{n}{(n-1)}}$.

In many measurement situations it is easier to envisage the standard deviation when it is expressed as a proportion of the mean. This statistic is called the coefficient of variation CV and is defined as

$$\mathrm{CV} = \frac{s(x)}{\bar{x}} \tag{3}$$

It can also be expressed as the percentage deviation or relative deviation RD,

$$\mathrm{RD} = \frac{100 s(x)}{\bar{x}}\% \tag{4}$$

Although the expression of the standard deviation in relative terms can be of value in many measurements, it should be recognized that some measurements, such as temperatures, have arbitrary zero points and these may lead to nonsensical relative deviations (see example 6).

Example 5: Consider the series of temperature measurements in example 2 with a mean value of 20 °C and a standard deviation of 0.9 °C. The relative deviation would then be (0.9/20) × 100 = 4.5 %. But if the data had been recorded in Kelvin, the mean value would have been (20 + 273.15) = 293.15 K and the standard deviation would have remained at 0.9 °C, giving a relative deviation of (0.9/293.15) × 100 = 0.31 %.

Example 6: The environmental test chamber of example 2 is used for a low temperature test and the temperature readings are

0.9 °C, –0.4 °C, 1.1 °C, –0.9 °C, 0.5 °C, 0.4 °C, 0.6 °C, 1.2 °C, –1.4 °C, –0.6 °C, –0.2 °C, –1.2 °C

With a mean of 0 °C and a standard deviation of 0.9 °C, the relative deviation would be 0.9/0 × 100 = ∞ %.

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8.2.3 *Standard uncertainty of a mean*

The standard deviation of the sample provides a statistical description of the variability of the data values within the sample and, by implication, an estimate of the variability of the data within any other sample drawn from the same overall population. So that, for example, if the cylinder of example 1 were to be re-measured across a different set of diameters one might expect the values to have roughly the same variability as those already recorded. However, the means of the two sets of data might reasonably be expected to vary to a much smaller extent and this is reflected in the statistical calculation of the standard deviation of the mean given by

$$s(\bar{x}) = \frac{s(x)}{\sqrt{n}} \tag{5}$$

The statistical dispersion of likely values for the mean defined by the standard deviation is then taken to be the standard uncertainty of the mean. Thus,

$$u(\bar{x}) = s(\bar{x}) \tag{6}$$

Example 7: The standard uncertainty of the mean diameter of the cylinder of example 1 is calculated as follows

$$\begin{aligned} u(\bar{x}) = s(\bar{x}) &= \frac{s(x)}{\sqrt{n}} \\ &= \frac{0.0103}{\sqrt{6}} \\ &= 0.0042 \ \text{mm} \end{aligned}$$

Example 8: The standard uncertainty of the mean test temperature of example 2 is calculated as follows

$$\begin{aligned} u(\bar{x}) = s(\bar{x}) &= \frac{s(x)}{\sqrt{n}} \\ &= \frac{0.905}{\sqrt{12}} \\ &= 0.261 \ ^{\circ}\text{C} \end{aligned}$$

8.2.4 *Standard uncertainty of a single value*

In some measurement situations there might only be a small number of observations of the measurand. This subclause deals with single observations; the treatment of small samples is dealt with in **8.2.8**. With a single observation, the evaluation of the uncertainty can be based on measurements taken earlier, provided the conditions of the measurement remain unchanged. It might be, for example, that a sample component is subjected to detailed measurement but that components from the production line are subjected to only a single measurement. In this case, the standard uncertainty is derived from the detailed measurement set as follows: the standard deviation of the set of measurements is obtained from equation (2) using n for the number of measurements in the set. The standard uncertainty of the mean is then obtained from equations (5) and (6) using $n = 1$ as the mean is that of a single value.

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Example 9: Having made the detailed set of measurements of cylinder diameter set out in example 1, the inspection manager decides that he will accept a larger uncertainty for an individual cylinder from the production line and orders that all production cylinders will be measured across only one diameter. The standard uncertainty of the diameter of the production cylinders is then calculated as follows:

$s(x)$ is obtained from the data of examples 1 and 3 as 0.010 3 mm and

$$u(\bar{x}) = s(\bar{x}) = \frac{s(x)}{\sqrt{n}}$$
$$= \frac{0.0103}{\sqrt{1}}$$
$$= 0.0103 \text{ mm}$$

Example 10: The research manager responsible for the environmental test chamber of example 2 decides that the effort involved in taking temperatures from the chamber at two-hourly intervals throughout the night is unjustified and that it will be sufficient to take a single reading of the temperature at the end of the evening shift. This increases the uncertainty of the test temperature, which is calculated as follows:

$s(x)$ is obtained from the data of examples 2 and 4 as 0.905 and

$$u(\bar{x}) = s(\bar{x}) = \frac{s(x)}{\sqrt{n}}$$
$$= \frac{0.905}{\sqrt{1}}$$
$$= 0.905 \ ^{\circ}\text{C}$$

8.2.5 *Degrees of freedom in a standard deviation*

As described in **8.2.2**, the standard deviation is a statistical description of the scatter of the data and, as with any description, the more data the description is based on the more reliable the description will be. The reliability of the standard deviation as a description of the scatter is defined by a statistic called the degrees of freedom, which is one less than the number of data points, thus

$$\nu = n - 1 \tag{7}$$

Example 11: The degrees of freedom in the standard deviation of the cylinder diameter measurements of example 3 are calculated as follows

$$\nu = n - 1$$
$$= 6 - 1$$
$$= 5$$

Example 12: The degrees of freedom in the standard deviation of the temperature measurements in the environmental chamber of example 4 are calculated as follows

$$\nu = n - 1$$
$$= 12 - 1$$
$$= 11$$

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8.2.6 *Expanded uncertainty of a sample mean*

Unless there is evidence to the contrary, it is common practice to assume that a series of experimental values will be scattered according to the normal or Gaussian distribution and, as will be seen in **8.3.2**, this implies that 68 % of the data will lie within one standard deviation of the mean. In the case of a sample mean, there is therefore an approximately one-in-three chance that the true mean will lie outside the uncertainty interval defined as one standard deviation either side of the sample mean. This is a level of risk that is unacceptable in almost all measurement situations. To increase the confidence that the true value will lie within the defined interval, that interval is increased by multiplying the standard uncertainty by a factor, k, based on the degrees of freedom in the standard deviation and on the level of confidence required. The resulting value is called the expanded uncertainty, U, and, in the case of a mean of a series of readings, is defined as follows:

$$U(\bar{x}) = ku(\bar{x}) \tag{8}$$

The value of the multiplier k is obtained from the Student's t table (Table 1) for the appropriate degrees of freedom and level of confidence.

Example 13: The inspection manager responsible for the manufacture of the cylinder of example 1 requires to know the expanded uncertainty of the mean diameter of the test cylinder at the 95 % confidence level. The standard uncertainty of the mean of the readings is 0.004 2 mm (example 7) and entering Table 1 with 5 degrees of freedom (see example 11) and 95 % confidence level, k = 2.57; the required expanded uncertainty is therefore given by

$$\begin{aligned} U(\bar{x}) &= ku(\bar{x}) \\ &= 2.57 \times 0.0042 \\ &= 0.0108 \ \text{mm} \end{aligned}$$

Example 14: The research manager responsible for operating the environmental chamber of example 2 requires to know the expanded uncertainty of the mean test temperature derived from the two-hourly readings at the 99 % confidence level. The standard uncertainty of the mean of the readings is 0.261 °C (example 8) and entering Table 1 with 11 degrees of freedom (see example 12) and 99 % confidence level, k = 3.11; the required expanded uncertainty is therefore given by

$$\begin{aligned} U(\bar{x}) &= ku(\bar{x}) \\ &= 3.11 \times 0.261 \\ &= 0.812 \ ^{\circ}\text{C} \end{aligned}$$

8.2.7 *Expanded uncertainty of a single value*

When the standard uncertainty is that of a single value and is derived from an earlier set of data, the standard uncertainty is calculated as set out in **8.2.4**, but the degrees of freedom associated with the standard deviation are those of the original set of data. It is this value of the degrees of freedom that is used in selecting the k factor from Table 1.

Example 15: Having decided to measure each cylinder across only one diameter, the inspection manager responsible for the manufacture of the cylinders of example 1 requires to know the expanded uncertainty of the diameter of the production cylinders at the 95 % confidence level. The standard uncertainty of the single reading is 0.010 3 mm (example 9) and entering Table 1 with five degrees of freedom (see example 11) and 95 % confidence level, k = 2.57; the required expanded uncertainty is therefore given by

$$\begin{aligned} U(\bar{x}) &= ku(\bar{x}) \\ &= 2.57 \times 0.0103 \\ &= 0.0265 \ \text{mm} \end{aligned}$$

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Example 16: The research manager responsible for operating the environmental chamber of example 2 requires to know the expanded uncertainty of the test temperature at the 95 % confidence level for those tests where the temperature is recorded only at the end of the evening shift. The standard uncertainty of the mean of the readings is 0.905 °C (example 10) and entering Table 1 with 11 degrees of freedom (see example 12) and 95 % confidence level, $k = 2.20$; the required expanded uncertainty is therefore given by

$$U(\bar{x}) = ku(\bar{x})$$
$$= 2.20 \times 0.905$$
$$= 1.99 \text{ °C}$$

Table 1 — Table of Student's t values

Degrees of freedom	Confidence level %		
	90	95	99
1	6.31	12.71	63.66
2	2.92	4.30	9.92
3	2.35	3.18	5.84
4	2.13	2.78	4.60
5	2.02	2.57	4.03
6	1.94	2.45	3.71
7	1.89	2.36	3.50
8	1.86	2.31	3.36
9	1.83	2.26	3.25
10	1.81	2.23	3.17
11	1.80	2.20	3.11
12	1.78	2.18	3.05
13	1.77	2.16	3.01
14	1.76	2.14	2.98
15	1.75	2.13	2.95
16	1.75	2.12	2.92
17	1.74	2.11	2.90
18	1.73	2.10	2.88
19	1.73	2.09	2.86
20	1.72	2.09	2.85
25	1.71	2.06	2.79
30	1.70	2.04	2.75
40	1.68	2.02	2.70
60	1.67	2.00	2.66
100	1.66	1.98	2.63
∞	1.64	1.96	2.58

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8.2.8 ***Reducing the expanded uncertainty by the use of past data***

Inspection of Table 1 shows that the value of the multiplier, k, used to derive the expanded uncertainty, depends strongly on the degrees of freedom associated with the standard deviation on which the standard uncertainty is based. The degrees of freedom can be increased, and k decreased, by using more data from past experience, and when several previous batches of data are available they may all be used to obtain the estimated standard deviation, provided that the measurements were taken under similar conditions. In this procedure, it is the variations about the mean of each set that are combined rather than the data sets themselves, and the method can still be employed if the batch means vary with time. It is, however, important that the dispersion of individual values about their batch mean remains essentially constant. The pooled standard deviation, s_p, is given by

$$s_p^2 = \frac{\sum\left\{(n_i - 1)s_i(x)^2\right\}}{\sum(n_i - 1)} \tag{9}$$

where n_i and s_i are the number of readings and the standard deviation of the i-th data set.

The degrees of freedom of a pooled standard deviation are the sum of the degrees of freedom of the pooled data sets. Thus,

$$\nu_p = \sum \nu_i = \sum(n_i - 1) \tag{10}$$

Example 17: The inspection manager responsible for the manufacture of the cylinders of example 1 recognizes that the expanded uncertainty of the production cylinders is strongly dependent on the high value of k that results from having only 5 degrees of freedom in the standard deviation. To reduce this value he decides to select two more cylinders at random from the production line and measure one of these across 2 diameters at three planes and the other across 3 diameters at three planes. The resulting values are

83.18 mm, 83.17 mm, 83.19 mm, 83.20 mm, 83.19 mm, 83.17 mm, and

83.16 mm, 83.15 mm, 83.18 mm, 83.17 mm, 83.16 mm, 83.16 mm, 83.17 mm, 83.18 mm, 83.17 mm

The standard deviations of the two new sets are calculated from equation (2) as 0.012 1 mm and 0.010 0 mm. These are pooled with the original set to give an overall standard deviation of

$$s_p = \sqrt{\frac{\sum\left\{(n_i - 1)s_i(x)^2\right\}}{\sum(n_i - 1)}}$$

$$= \sqrt{\frac{(6-1)0.0103^2 + (6-1)0.0121^2 + (9-1)0.0100^2}{(6-1)+(6-1)+(9-1)}}$$

$$= 0.0107 \text{ mm}$$

The total degrees of freedom are given by

$$\nu_p = \sum \nu_i = \sum(n_i - 1)$$
$$= (6-1)+(6-1)+(9-1)$$
$$= 18$$

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For 18 degrees of freedom and 95 % confidence Table 1 gives k as 2.10, and the expanded uncertainty of the diameter of the production cylinders is then

$$U(\bar{x}) = ku(\bar{x})$$
$$= 2.10 \times 0.0107$$
$$= 0.0225 \text{ mm}$$

8.3 Type B evaluation of uncertainties

8.3.1 *Overview of the Type B evaluation method*

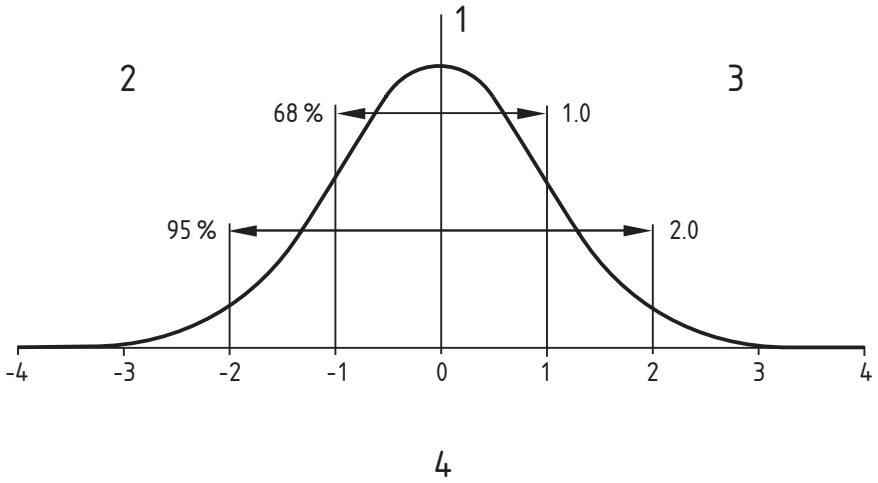
While the Type A evaluation of uncertainty set out in the preceding subclause provides a measure of the uncertainty due to random variations in the measurand, it cannot readily be applied to the analysis of such parameters as the calibration of an instrument or the drift between calibrations. Such sources of uncertainty should instead be evaluated on the basis of the information available, which may come from a calibration certificate, manufacturers' specifications or professional judgement and past experience. These non-statistical sources of information lead to a Type B uncertainty evaluation.

Perhaps the most important difference between Type A and Type B evaluations of uncertainty is that, whereas the standard deviation of the Type A evaluation specifies an interval within which the measurand is expected to lie with a defined level of confidence, in a Type B evaluation the instrumentation expert has to specify not only the interval but also the level of confidence that the value of the measurand will lie within the defined interval, and whether it is more likely to lie in one region of the interval than in another. Having made these decisions, it is necessary to reduce the interval to a standard uncertainty so that the component uncertainties are brought to a common basis for combination.

Fortunately, the vast majority of measurement situations can be adequately represented by a very limited number of distributions and for these the standard uncertainty can be calculated easily.

8.3.2 *Normal distribution*

The normal distribution, for which a plot of probability density against value results in the bell-shaped curve of Figure 1, is the commonest distribution for a set of experimental readings obtained for a Type A evaluation.



Key
1 Probability density
2 Percent of readings in interval
3 Intervals – standard deviations either side of mean
4 Standard

Figure 1 — The standard normal, or Gaussian, distribution

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However, this distribution is also encountered in Type B evaluations for such sources as calibration uncertainties where the expanded uncertainty at a quoted level of confidence will usually be obtained from an evaluation of the standard uncertainty of the calibration, based on the contributing sources, and a coverage factor based on an assumption of the normal distribution. If the calibration certificate quotes the coverage factor used, the calculation of the standard uncertainty is straightforward but if the certificate quotes a confidence level, it is necessary to obtain the value of k from Table 1; it is assumed, unless otherwise stated, that the degrees of freedom will be infinite. The standard uncertainty can then be calculated from the expanded uncertainty from

$$u(x_i) = \frac{U(x_i)}{k} \tag{11}$$

Example 18: A voltmeter is calibrated by a UKAS-accredited calibration laboratory. The calibration certificate states that

"the expanded uncertainty of the calibration is 0.1 % of full scale reading over the range 1 to 100 volts at the 95 % confidence level."

Although the uncertainty is quoted in percentage terms it should be noted that this is a percentage of full scale (or 100 volts). The expanded uncertainty is therefore 0.1 × 100/100 = 0.1 volts throughout the calibrated range.

The uncertainty statement requires the user to make an assumption about the underlying distribution in order to determine the k factor: in the absence of any further information it is common practice to assume a normal distribution. Since the stated confidence level is 95 % Table 1 gives a value of k = 1.96 and the standard uncertainty is

$$\begin{aligned} u(x_i) &= \frac{U(x_i)}{k} \\ &= \frac{0.1}{1.96} \\ &= 0.05 \text{ V} \end{aligned}$$

Example 19: A digital bore micrometer is supplied with a calibration certificate stating that

"the measurement uncertainty is 0.005 mm (k = 2) over the range 100 to 125 mm."

As the k factor is quoted in the uncertainty statement no assumption about the distribution is required. The standard uncertainty is therefore

$$\begin{aligned} u(x_i) &= \frac{U(x_i)}{k} \\ &= \frac{0.005}{2} \\ &= 0.0025 \text{ mm} \end{aligned}$$

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Example 20: A laboratory pH meter is supplied with a calibration certificate stating

"the measurement uncertainty is 0.02 pH at a 99 % confidence level."

As with example 19, this uncertainty statement requires its user to make an assumption about the distribution and again the assumption is that the normal distribution will apply. For 99 % confidence Table 1 gives $k = 2.58$ and the standard uncertainty is therefore

$$u(x_i) = \frac{U(x_i)}{k}$$
$$= \frac{0.02}{2.58}$$
$$= 0.0078 \text{ pH}$$

8.3.3 *Rectangular distribution*

The rectangular distribution (Figure 2) applies to those sources of uncertainty where the true value can lie anywhere within a specified range with equal probability. Unlike the normal distribution, the limits of the rectangular distribution provide 100 % coverage. The standard uncertainty can be shown to be given by

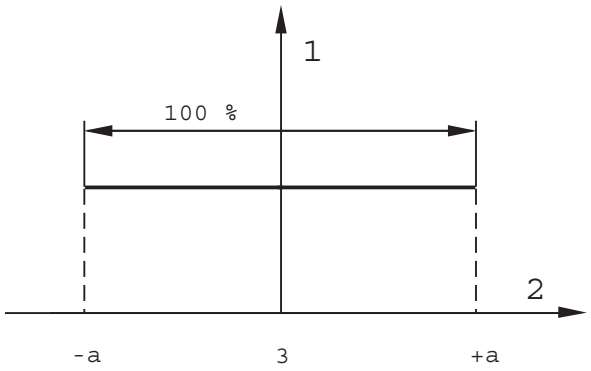
$$u(x_i) = \frac{a}{\sqrt{3}} \tag{12}$$

where a is the semi-width of the distribution.

The two most common situations represented by the rectangular distribution are instrument and data acquisition resolution, and acceptance criteria in calibration.

In the first of these, a reading on a digital display could represent any value of the measurand that lies between the smallest value that gives the displayed output rather than one digit less and the largest value that gives the displayed value rather than one digit greater. There is no way of determining where in this range the true value lies and all values must therefore have the same probability. The expanded uncertainty is half this range and is therefore half the least significant digit in the display.

In addition to digitally displayed outputs the effect is seen in data logged by computer, with the resolution of the analogue-to-digital converter contributing to the uncertainty of the measurement. The limits in this case are half the value represented by one bit of the digital value.



Key

1 Probability density

2 Quantity X_i

Figure 2 — Rectangular distribution

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It is common practice in metrology to accept an instrument's output without correction provided the meter error in the calibration falls within some specified limit, the acceptability criterion. This practice gives rise to an uncertainty with a rectangular distribution since the meter error can lie anywhere within the specified range with equal probability.

Example 21: The pH meter of example 20 displays the pH value to 0.01 pH.

The limits due to the resolution of the display are therefore 0.01/2 = 0.005 pH and the standard uncertainty is

$$u(x_i) = \frac{a}{\sqrt{3}} = \frac{0.005}{\sqrt{3}} = 0.0029 \text{ pH}$$

Example 22: The output of the voltmeter of example 18 is to be recorded by a computer data acquisition system. The analogue-to-digital converter is a 12-bit device giving a full-range resolution of 1 part in 2^{12} = 4 096 and this is set up to cover the full 0 to 100 volt output of the meter.

A single bit on the A-D converter then represents 100/4 096 = 0.024 V and the resolution limit is 0.024/2 = 0.012 V.

The standard uncertainty is thus

$$u(x_i) = \frac{a}{\sqrt{3}} = \frac{0.012}{\sqrt{3}} = 0.0070 \text{ V}$$

Example 23: The digital bore micrometer of example 19 is checked on a weekly basis in the Production Inspection Department by measuring the diameter of a ring gauge. Provided that this weekly measurement lies within 0.01 mm of the ring gauge certificate value, the bore micrometer is used without correction. If the measurement is outside this range, the instrument is returned to the manufacturer for adjustment and re-calibration.

The check measurement can lie in the range 0.01 mm either side of the certificate value and the acceptance limit is therefore 0.01 mm. As any value in the range is acceptable, this source of uncertainty is assumed to have a rectangular distribution and the standard uncertainty arising from the acceptance criterion is therefore

$$u(x_i) = \frac{a}{\sqrt{3}} = \frac{0.01}{\sqrt{3}} = 0.0058 \text{ mm}$$

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8.3.4 *Triangular distribution*

The rectangular distribution can be applied to any quantities where it is only possible to set limits but where there is no information on the likely distribution of possible values. There might be cases where there is good reason to believe that values close to the assigned limits are less likely than those close to the centre of the range. In these circumstances, the rectangular distribution would give too pessimistic a value for the standard uncertainty and it might be more realistic to assume a triangular distribution, Figure 3. For this, the standard uncertainty is given by

$$u(x_i) = \frac{a}{\sqrt{6}} \tag{13}$$

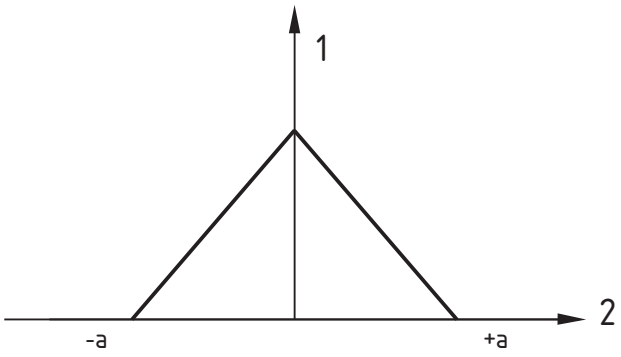
where a is the semi-width of the distribution.

8.3.5 *Two-valued distribution*

Some instruments are affected by hysteresis or internal friction and this results in a quantity that takes one of two values (Figure 4). In this case the standard uncertainty is given by

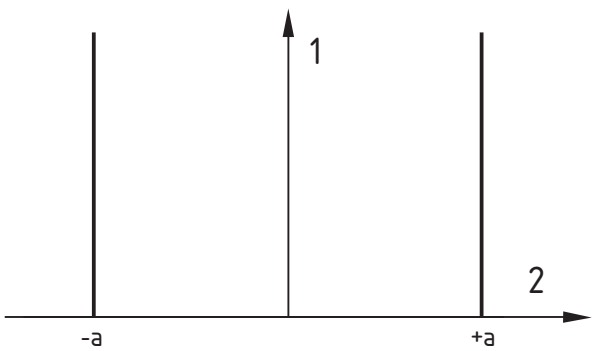
$$u(x_i) = a \tag{14}$$

where a is half the interval between the two possible values.



Key
1 Probability density
2 Quantity X_i

Figure 3 — Triangular distribution



Key
1 Probability density
2 Quantity X_i

Figure 4 — Two-valued distribution

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8.3.6 *Handling asymmetry*

The upper and lower bounds of a distribution might in some circumstances not be symmetrical about the best estimate (Figure 5) and, in extreme cases, the interval might even be distributed entirely to one side of a limiting value. For example, in measuring the height of a storage vessel any deviation of the measurement from the vertical will result in an over-estimation of the true height.

The GUM illustrates an approximate method for handling this type of uncertainty using a rectangular distribution (GUM **4.3.8**). The distribution is assigned a full range equal to the range from the lower to the upper bound and the standard uncertainty is then given by

$$u(x_i) = \frac{b_1 + b_2}{\sqrt{12}} \tag{15}$$

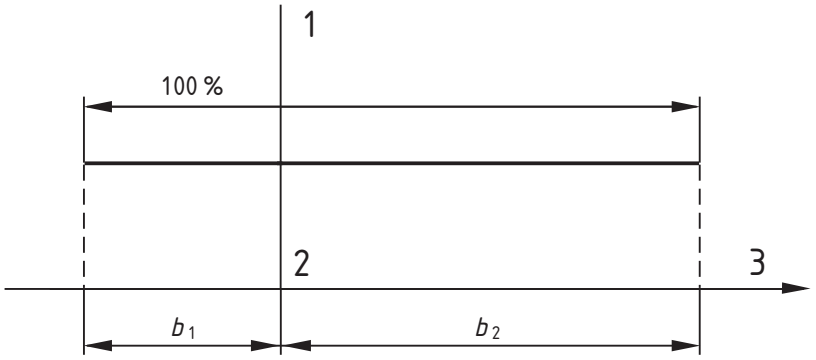
where b_1 and b_2 are defined in Figure 5.

This approach underestimates the uncertainty on the side with the larger bound and a more conservative approach would be to assume a distribution based entirely on the larger bound, giving

$$u(x_i) = \text{greater of } \frac{b_1}{\sqrt{3}} \text{ or } \frac{b_2}{\sqrt{3}} \tag{16}$$

A similar approach can be applied based on a triangular distribution when this is deemed appropriate.

If the asymmetric element of uncertainty is large in comparison with the uncertainty due to other sources, it might be appropriate to perform two analyses and calculate two values for the combined uncertainty, one based on the larger asymmetric bound and the other on the smaller. The two values would then be quoted in the uncertainty statement.



Key

1 Probability density
2 Mean
3 Quantity X_i

Figure 5 — Asymmetric distribution

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9 Estimating the effect of input uncertainties on the result

9.1 Sensitivity coefficients

The input quantities, X_i, to a measurement can contribute in different ways to the output quantity, Y, and it would be inappropriate to consider the uncertainties in the input quantities without considering the impact of each input quantity on the final measurement result. Consider, for example, the volume of liquid stored in a cylindrical vessel standing with its axis vertical. The volume is given by

$$V = \frac{\pi d^2 h}{4}$$

where

d is the internal diameter of the vessel; and

h is the depth of the liquid.

Rewriting this as

$$V = \frac{\pi d d h}{4}$$

it can be seen that the diameter appears twice and, if the uncertainties are of similar magnitude, uncertainties in the diameter might therefore be expected to contribute more to the uncertainty in the volume than those in the depth, which only appears once.

Before the input uncertainties can be combined they should be translated into the resulting uncertainties in the output quantity. The relationship between an input uncertainty and its contribution to the uncertainty in the output quantity is given by the sensitivity coefficient. Defined as the partial derivative, $\partial f/\partial x_i$, of the functional relationship, f, between the input quantities, $x_1 \ldots x_N$, and the output quantity, y, evaluated at the nominal values of the input quantities, the sensitivity coefficient can be obtained in two ways, analytically or numerically.

Just as uncertainties can be expressed in absolute or relative terms, so sensitivity coefficients can be absolute or relative, and it is important to ensure that absolute coefficients are used with absolute uncertainties and relative coefficients with relative uncertainties.

9.2 Deriving the sensitivity coefficients analytically

From its definition as the partial derivative of the output estimate y with respect to the input estimate x_i at the nominal value of all the input quantities $x_1 \ldots, x_N$, the sensitivity coefficient c_i is given by

$$c_i = \frac{\partial f}{\partial x_i} \tag{17}$$

The relative sensitivity coefficient c^*_i is then given by

$$c_i^* = \frac{x_i}{y}\frac{\partial f}{\partial x_i} \tag{18}$$

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Example 24: The volume of liquid in an upright cylindrical vessel is given by

$$V = \frac{\pi d^2 h}{4}$$

where

d is the diameter equal to 2.100 m; and

h is the height equal to 3.600 m

The sensitivity coefficient for d is given by

$$c_d = \frac{\partial V}{\partial d} = \frac{2\pi dh}{4} = \frac{\pi dh}{2} = \frac{\pi \times 2.1 \times 3.6}{2} = 11.88 \text{ mm}$$

and the relative sensitivity coefficient for d is given by

$$c_d^* = \frac{d}{V}\frac{\partial V}{\partial d} = \frac{4d}{\pi d^2 h}\frac{2\pi dh}{4} = 2$$

The sensitivity coefficient for h is given by

$$c_h = \frac{\partial V}{\partial h} = \frac{\pi d^2}{4} = \frac{\pi(2.1)^2}{4} = 3.46 \text{ mm}$$

and the relative sensitivity coefficient for h is given by

$$c_h^* = \frac{h}{V}\frac{\partial V}{\partial h} = \frac{4h}{\pi d^2 h}\frac{\pi d^2}{4} = 1$$

Example 25: The torsional stiffness of a solid shaft is given by

$$S = \frac{\pi d^4 G}{32L}$$

where

d is the diameter equal to 0.050 m

G is the modulus of rigidity equal to 8×10^{10} Pa; and

L is the length equal to 0.750 m

The sensitivity coefficient for d (N/rad) is given by

$$c_d = \frac{\partial S}{\partial d} = \frac{4\pi d^3 G}{32L} = \frac{4\pi 0.050^3 \times 8 \times 10^{10}}{32 \times 0.750} = 5236000$$

and the relative sensitivity coefficient for d is given by

$$c_d^* = \frac{d}{S}\frac{\partial S}{\partial d} = \frac{32Ld}{\pi d^4 G}\frac{4\pi d^3 G}{32L} = 4$$

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The sensitivity coefficient for G (m^3) is given by

$$c_G = \frac{\partial S}{\partial G} = \frac{\pi d^4}{32L} = \frac{\pi 0.050^4}{32 \times 0.750} = 8.181 \times 10^{-7}$$

and the relative sensitivity coefficient for G is given by

$$c_G^* = \frac{G}{S}\frac{\partial S}{\partial G} = \frac{32LG}{\pi d^4 G}\frac{\pi d^4}{32L} = 1$$

The sensitivity coefficient for L (N·m/rad) is given by

$$c_L = \frac{\partial S}{\partial L} = -\frac{\pi d^4 G}{32L^2} = -\frac{\pi 0.050^4 \times 8 \times 10^{10}}{32 \times 0.750^2} = -87266$$

and the relative sensitivity coefficient for L is given by

$$c_L^* = \frac{L}{S}\frac{\partial S}{\partial L} = \frac{32L \cdot L}{\pi d^4 G}\left(-\frac{\pi d^4 G}{32L^2}\right) = -1$$

Examination of the relative sensitivity coefficients calculated in examples 24 and 25 shows that, in the particular case of functional relationships that take the form of simple products, the relative sensitivity coefficient for each input quantity is given by the power applied to that quantity in the functional relationship.

9.3 Deriving the sensitivity coefficients numerically

While the analytical method of deriving the sensitivity coefficients is straightforward for simple relationships, it can quickly become unmanageable for more complex relationships. The alternative approach is to obtain the sensitivity coefficients by a simple numerical technique that can be coded into a spreadsheet or other software for almost any defined relationship (see **B.1**).

The technique involves calculating the output estimate y for two possible values of the input quantity X_i and deriving the sensitivity as follows. If the result of the calculation using an input value of $x_i - \Delta x_i$ is y_- and using $x_i + \Delta x_i$ is y_+ then

$$c_i = \frac{y_+ - y_-}{2\Delta x_i} \tag{19}$$

and

$$c_i^* = \frac{x_i}{y}\frac{(y_+ - y_-)}{2\Delta x_i} \tag{20}$$

The choice of the increment Δx_i is important. The technique calculates the mean gradient of the functional relationship over the interval $2\Delta x_i$, and if the functional is non-linear the use of too large a value of Δx_i will give a wrong value for c_i. Choosing too small a value can, in some circumstances, result in problems with round-off error in the calculation. It is also necessary to retain enough digits at each stage of the calculation to ensure that c_i is calculated with sufficient accuracy. The value of c_i of interest is that due to the effect of the uncertainty in x_i and the recommendation is that the value chosen for Δx_i should be close to the standard uncertainty of the input.

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Example 26: The sensitivity coefficients for the diameter and depth of the cylindrical vessel of example 24 can be calculated numerically as follows:

Volume for diameter 2.100 m and depth 3.600 m = 12.468 98 m^3

Assume an increment of 1 mm in both depth and diameter

Volume for diameter 2.099 m and depth 3.600 m = 12.457 11 m^3

Volume for diameter 2.101 m and depth 3.600 m = 12.480 86 m^3

Increment in volume = 12.480 86 – 12.457 11 = 0.023 75 m^3

Increment in diameter = 0.002 m

The sensitivity coefficient for diameter is then given by equation (19)

$$c_d = \frac{\Delta V}{\Delta d} = \frac{0.02375}{0.002} = 11.88 \ \ \mathrm{m}^2$$

and the relative sensitivity coefficient is given by equation (20)

$$c_d^* = \frac{d}{V}\frac{\Delta V}{\Delta d} = \frac{2.1}{12.46898}\frac{0.02375}{0.002} = 2.000$$

Volume for diameter 2.100 m and depth 3.599 m = 12.465 52 m^3

Volume for diameter 2.100 m and depth 3.601 m = 12.472 44 m^3

Increment in volume = 12.472 44 – 12.465 52 = 0.006 92 m^3

Increment in diameter = 0.002 m

The sensitivity coefficient for depth is then given by equation (19)

$$c_h = \frac{\Delta V}{\Delta h} = \frac{0.00692}{0.002} = 3.46 \ \ \mathrm{m}^2$$

and the relative sensitivity coefficient is given by equation (20)

$$c_h^* = \frac{h}{V}\frac{\Delta V}{\Delta h} = \frac{3.6}{12.46898}\frac{0.00692}{0.002} = 0.999$$

The minor difference in the value for c_h* from that obtained in example 24 is due to round-off in the calculation in the numerical method.

Example 27: The sensitivity coefficients for the diameter, modulus of rigidity and length of the solid shaft of example 25 can be calculated numerically as follows:

Assume an increment of 0.05 mm in diameter, 0.5×10^9 Pa in modulus of rigidity and 0.5 mm in length

Torsional stiffness for nominal values = 65 449.9 N·m/rad

Torsional stiffness for 50.05 mm diameter = 65 712.0

Torsional stiffness for 49.95 mm diameter = 65 188.4

Increment in stiffness = 65 712.0 – 65 188.4 = 523.6 N·m/rad

Increment in diameter = 0.000 1 m

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The sensitivity coefficient for diameter (N/rad) is then given by equation (19)

$$c_d = \frac{\Delta S}{\Delta d} = \frac{523.6}{0.0001} = 5236000$$

and the relative sensitivity coefficient is given by equation (20)

$$c_d^* = \frac{d}{S}\frac{\Delta S}{\Delta d} = \frac{0.05}{65449.8}\ \frac{523.6}{0.0001} = 4.000$$

Torsional stiffness for modulus of rigidity 80.5×10^9 Pa = 65 858.9 N·m/rad
Torsional stiffness for modulus of rigidity 79.5×10^9 Pa = 65 040.8 N·m/rad
Increment in stiffness = 65 858.9 – 65 040.8 = 818.1 N·m/rad
Increment in modulus of rigidity = 1.0×10^9 Pa

The sensitivity coefficient for modulus of rigidity is then given by equation (19)

$$c_G = \frac{\Delta S}{\Delta G} = \frac{818.1}{1.0 \times 10^9} = 8.181 \times 10^{-7}$$

and the relative sensitivity coefficient is given by equation (20)

$$c_L^* = \frac{G}{S}\frac{\Delta S}{\Delta G} = \frac{80 \cdot 10^9}{65449.8}\ \frac{818.1}{1 \times 10^9} = 1.000$$

Torsional stiffness for length of 0.750 5 m = 65 406.2 N·m/rad
Torsional stiffness for length of 0.749 5 m = 65 493.5 N·m/rad
Increment in stiffness = 65 406.2 – 65 493.5 = –87.3 N·m/rad
Increment in length = 0.001 m

The sensitivity coefficient for length (N·m/rad) is then given by equation (19)

$$c_L = \frac{\Delta S}{\Delta L} = \frac{-87.3}{0.001} = -87300$$

and the relative sensitivity coefficient is given by equation (20)

$$c_L^* = \frac{L}{S}\frac{\Delta S}{\Delta L} = \frac{0.750}{65449.8}\ \frac{-87.3}{0.001} = -1.000$$

9.4 Contribution to uncertainty

Having obtained the sensitivity coefficients, the next step is, for $i = 1, \ldots, N$, to calculate the contribution, $u_i(y)$, that the uncertainty $u(x_i)$ makes to the uncertainty in the final result. This is defined as the component of the combined standard uncertainty $u_c(y)$ of the output estimate y generated by the standard uncertainty $u(x_i)$ of the input estimate x_i and is given by equation (21), or in relative form by equation (22).

$$u_i(y) = c_i u(x_i) \tag{21}$$

$$u_i^*(y) = c_i^* u^*(x_i) \tag{22}$$

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Example 28: If the standard uncertainties of the vessel diameter and liquid depth in examples 24 and 26 are 1 mm = 0.001 m, the contributions to volume uncertainty are given, in absolute terms (m^3), by equation (21) as

$$u_d(V) = c_d u(d) = 11.875 \times 0.001 = 0.0119$$

$$u_h(V) = c_h u(h) = 3.464 \times 0.001 = 0.00346$$

When working in relative terms the relative uncertainty in diameter is 0.001/2.100 = 0.000 476 and in depth is 0.001/3.6 = 0.000 278. Equation (22) then gives the contributions to relative uncertainty in the volume as

$$u_d^*(V) = c_d^* u^*(d) = 2 \times 0.000476 = 0.000952$$

$$u_h^*(V) = c_h^* u^*(h) = 1 \times 0.000278 = 0.000278$$

Example 29: If the standard uncertainties of the shaft diameter, modulus of rigidity and shaft length in examples 25 and 27 are 0.5 mm = 0.000 5 m, 2×10^9 Pa, and 1 mm = 0.001 m respectively the contributions to the uncertainty of the torsional stiffness are given, in absolute terms (N·m/rad), by equation (21) as

$$u_d(S) = c_d u(d) = 5235988 \times 0.0005 = 2618$$

$$u_G(S) = c_G u(G) = 8.1812 \times 10^{-7} \times 2 \times 10^9 = 1636.41$$

$$u_L(S) = c_L u(L) = 87266.5 \times 0.001 = 87.26$$

When working in relative terms, the relative uncertainty in diameter is 0.000 5/0.050 = 0.01, in modulus of rigidity is $2 \times 10^9 / 80 \times 10^9 = 0.025$ and in length is 0.001/0.750 = 0.001 333. Equation (22) then gives the contributions to relative uncertainty in the torsional rigidity as

$$u_d^*(S) = c_d^* u^*(d) = 4 \times 0.01 = 0.04$$

$$u_G^*(S) = c_G^* u^*(G) = 1 \times 0.025 = 0.025$$

$$u_L^*(S) = c_L^* u^*(L) = -1 \times 0.001333 = -0.001333$$

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10 Combining the input uncertainties

10.1 Defining correlations between input quantities

Before combining the contributions from the various sources of uncertainty, it is important to consider whether or not there are any links between those sources. Where sources are linked their uncertainties should be treated differently from those of unlinked sources.

Consider the measurement of the volume of a cube. If the same tape rule is used for all three measurements, any error in the calibration of the rule will be transferred to all three measurements in the same way and the sources of uncertainties due to calibration will be linked. If the rule measures short, it will under-read all three measurements and the measurements are said to be correlated. However, each measurement will also be subject to a resolution uncertainty and these sources of uncertainties will not be linked since each is essentially random; such sources of uncertainty are described as uncorrelated.

Correlated quantities will either all be high at the same time or all be low at the same time; however, it is unlikely that all uncorrelated quantities will be high, or low, at the same time.

Examples of likely correlated sources are:

a) the use of the same instrument to perform more than one measurement, because this will result in the calibration sources of uncertainty being correlated;

b) environmental effects: readings affected by ambient conditions of temperature and humidity will experience the same influence over a wide area of a process plant unless shielded from the environment.

Examples of uncorrelated sources are:

a) resolution and data acquisition effects, including those data acquisition uncertainties arising from analogue-to-digital converter resolution; and

b) acceptance criteria in calibration.

10.2 Combining uncertainties in uncorrelated input quantities

The method of combining uncertainties associated with uncorrelated effects recognizes that they are unlikely all to be high (or low) simultaneously and that it is therefore likely that the combined uncertainty will be less than the sum of the individual sources. The combination is by the root-sum-square technique (GUM **5.1.2**) and the combined uncertainty is given by

$$u_{\mathrm{c}}(y) = \sqrt{\sum_{i=1}^{N} u_i(y)^2} = \sqrt{\sum_{i=1}^{N} \left(c_i u(x_i)\right)^2} \tag{23}$$

or in relative terms

$$u_{\mathrm{c}}^{*}(y) = \sqrt{\sum_{i=1}^{N} u_i^{*}(y)^2} = \sqrt{\sum_{i=1}^{N} \left(c_i^{*} u^{*}(x_i)\right)^2} \tag{24}$$

It is vitally important to remember that absolute and relative uncertainties cannot be mixed and that therefore either all contributions should be combined using absolute sensitivity coefficients and source uncertainties, equation (23), or all should be combined using relative sensitivity coefficients and source uncertainties, equation (24).

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10.3 Combining uncertainties in correlated input quantities

In recognition of the fact that, being correlated, they will all be high (or low) simultaneously and that the combined uncertainty will therefore be equal to the sum of the component uncertainties, the summation of correlated inputs is obtained by the simple arithmetic summation of the individual correlated contributions (see Note to GUM **5.2.2**). Thus

$$u_{\mathrm{c}}(y) = \sum_{i=1}^{N} u_i(y) = \sum_{i=1}^{N} \left(c_i u(x_i) \right) \tag{25}$$

or in relative terms

$$u_{\mathrm{c}}^{*}(y) = \sum_{i=1}^{N} u_i^{*}(y) = \sum_{i=1}^{N} \left(c_i^{*} u^{*}(x_i) \right) \tag{26}$$

In applying equations (25) and (26), it is important that the sign of the individual sensitivity coefficients, c_i or c^*_i, is retained to ensure that the impact of the correlation is properly reflected in the summation. Again it is vital to realize that the summation should be either entirely in absolute or entirely in relative terms.

10.4 Handling combinations of correlated and uncorrelated uncertainties

While all sources in a measurement system could be uncorrelated it is highly unlikely that they will all be correlated. Where some sources are correlated and others uncorrelated, the combination proceeds as follows.

a) Identify the correlated sources and sum these arithmetically. There might be several groups each correlated by a different mechanism; each group is then summed separately using equation (25) or (26) as appropriate.

b) The results of the different correlated-group summations are then uncorrelated with each other and their combination with the remaining, wholly uncorrelated, sources of uncertainty proceeds using equation (23) or (24) as appropriate.

The preceding subclause deals with components of measurement that are either wholly correlated or wholly uncorrelated. In some measurement situations, the components might be partially correlated. In these circumstances, the analysis is more complicated (GUM **5.2**) and the best approach is to perform two analyses, the first on the basis of the relevant components of the measurement being wholly uncorrelated and the second on the basis of them being wholly correlated. A comparison of the two answers will then show whether the extent of the correlation is important or not. If this approach suggests that the overall uncertainty is sensitive to the assumptions regarding correlation, **5.2** of the GUM should be consulted.

11 Expressing the result

11.1 Coverage factors

In Clause **8**, all input uncertainties were reduced to standard uncertainties to provide a common basis for summation. The resulting combined uncertainty obtained from equations (23) to (26) is therefore also a standard uncertainty. It was seen in Clause **8** that the interval defined by one standard uncertainty either side of the mean covers only a limited range of the likely spread of values. This coverage is insufficient for most measurement purposes and a coverage of 95 % is normally required. **8.2.6** demonstrated the technique for obtaining an expanded uncertainty, with the required coverage, from a standard uncertainty when dealing with an input uncertainty obtained from a Type A evaluation. The same technique applies equally well to the calculation of an expanded combined uncertainty in the final result. Thus the expanded uncertainty at the required level is obtained from

$$U(y) = ku(y) \tag{27}$$

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or in relative terms

$$U^*(y) = ku^*(y) \tag{28}$$

where k in each case is selected from Table 1 for the required level of confidence, or coverage.

In selecting the value of k the probability distribution for the measurand Y is considered to follow the normal distribution even though many of the input uncertainties might have been assessed as rectangular. The reasoning behind this is that it can be shown that when several input quantities that follow rectangular distributions of the same or comparable width are summed, the result tends quickly (i.e. as the number of quantities increases) to a normal distribution. It can also be shown that a similar result is obtained when other distributions are combined. In selecting a k value from Table 1, it is also common practice to assume that the degrees of freedom in the final result are infinite as each input uncertainty contributes its own degrees of freedom resulting in a sufficiently large value for the final output for the assumption of infinity to hold with sufficient accuracy.

When the output uncertainty is dominated by a single input source that is derived from an evaluation of uncertainty based on very few readings or on a Type B evaluation where the limits are not defined with any degree of precision, it is necessary to adopt a more rigorous approach to the selection of the k factor. The approach is described in detail in the GUM Annex G.

11.2 The uncertainty statement

The purpose of the uncertainty evaluation is to define the result of the measurement in terms of three parameters: the mean value (the output estimate y), the expanded uncertainty $U(y)$, and the confidence level or coverage factor k. For the result to be meaningful all three values should be stated and an uncertainty statement will typically be of the form:

— "The result of the measurement is y units with an expanded uncertainty at the 95 % confidence level of $U(y)$ units"; or

— "The result of the measurement is y units with an expanded uncertainty (k = 2) of $U^*(y)$ %".

In defining the coverage factor k to obtain the expanded uncertainty, it is acceptable to state either the confidence level as a percentage or the coverage factor as a k value. When the confidence level is quoted, it is implied that a normal distribution with infinite degrees of freedom has been assumed.

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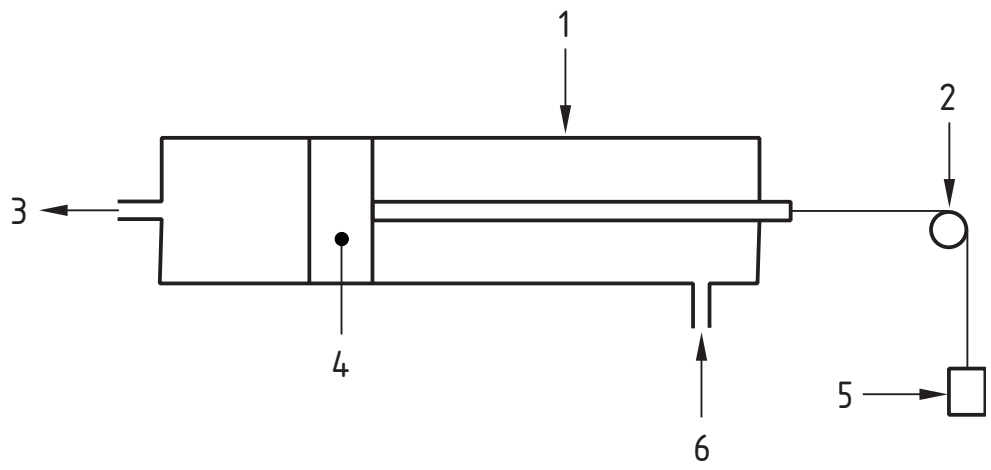
Annex A (informative)
Example of an uncertainty analysis

A.1 General

This example considers a theoretical piston prover used to calibrate liquid flowmeters. Flow is passed into a finely finished cylinder upstream of a piston. The flow carries the piston down the cylinder, expelling fluid from the outlet and through the flowmeter being calibrated. The quantity of fluid passing through the flowmeter is therefore measured by the travel of the piston combined with the cross-sectional area of the cylinder. In effect, after corrections, the volume of fluid passing through the meter is the volume displaced from the cylinder. The travel of the piston is measured using a rotary encoder driven by the piston movement. In this example, when the piston reaches the end of its travel, the flow is stopped and the piston returned to the start position. See Figure A.1.



Key
1 Cylinder
2 Encoder
3 Flow out
4 Piston
5 Tension weight
6 Flow in

Figure A.1 — Piston prover

A.2 Governing equation

$$\begin{aligned} Q &= \frac{AL}{t} \\ &= \frac{\pi d^2 L}{4t} \\ &= \frac{\pi d^2 mP}{4t} \end{aligned} \qquad \text{(A.1)}$$

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where

Q is the volume of fluid displaced by the piston;

A is the cylinder cross-sectional area;

d is the cylinder diameter;

L is the distance moved by the piston, equal to mP;

m is the movement per pulse;

P is the number of pulses counted; and

t is the time taken to count P pulses.

The cylinder has a nominal diameter of 75 mm, a minimum of 5 000 pulses are collected in any test run and the minimum time of passage is 20 s. A full traverse of the piston covers 400 mm and results in 20 000 pulses, giving a nominal value for m of 0.02 mm/pulse.

A.3 Contributory uncertainties

A.3.1 *General*

Uncertainties arise in the determination of d, m, P and t, and, as these four measurements are uncorrelated, the application of equation (23) yields

$$u_c^2(Q) = c_d^2 u^2(d) + c_m^2 u^2(m) + c_P^2 u^2(P) + c_t^2 u^2(t) \qquad \text{(A.2)}$$

A.3.2 *Uncertainties in diameter*

Uncertainties in the diameter arise from three sources:

a) variations in the cylinder diameter due to out of roundness;

b) variations in the cylinder diameter along its length; and

c) the uncertainty of the calibration of the measuring instrument used to determine the diameter.

The first two sources are evaluated by measuring the diameter across three different diameters at each of four planes along the cylinder length. The values obtained in this exercise are given in Table B.1 and the metrology laboratory's calibration certificate states that "the expanded uncertainty (k = 2) of the diameter determination is ±0.002 mm".

A check test on the diameter measurements should be carried out at this point to show that they are random along the length, that the tube is not tapered and that it does not have significant differences in area along the length. A taper or wavy profile could lead to systematic differences in volume along the length. If this were the case, it would be necessary either to apply a correction or to increase the uncertainty.

In the example this check is assumed to give a random variation of diameter uncertainty.

Table A.1 — Measured diameters at various planes

Plane	Diameter mm		
1	75.003	75.005	74.999
2	75.004	74.997	75.002
3	74.996	75.004	75.002
4	75.004	74.998	75.001

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The uncertainty due to variations in the diameter is obtained from a Type A evaluation of the measurements in Table A.1 as follows.

The mean diameter is given by equation (1) as

$$\bar{d} = \frac{d_1 + d_2 + \ldots + d_n}{n}$$
$$= \frac{75.003 + 75.005 + 74.999 + \ldots + 75.004 + 74.998 + 75.001}{12}$$
$$= 75.00125 \text{ mm}$$

The standard deviation of the measurements is given by equation (2) as

$$s_{\mathrm{d}} = \sqrt{\frac{(d_1 - \bar{d})^2 + (d_2 - \bar{d})^2 + \ldots + (d_n - \bar{d})^2}{(n-1)}}$$

$$= \sqrt{\frac{(75.003 - 75.00125)^2 + (75.005 - 75.00125)^2 + \ldots + (75.001 - 75.00125)^2}{(12-1)}}$$

$$= 0.003049$$

and the standard uncertainty of the mean is then given by equation (5) as

$$u_{\mathrm{d}} = \frac{s_{\mathrm{d}}}{\sqrt{n}}$$
$$= \frac{0.003049}{\sqrt{12}}$$
$$= 0.000880$$

The standard uncertainty associated with the calibration laboratory is obtained from equation (11) as

$$u_{\mathrm{cal}} = \frac{U_{\mathrm{cal}}}{k} = \frac{U_{\mathrm{cal}}}{2} = \frac{0.002}{2} = 0.001 \text{ mm}$$

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The uncertainties due to the variation in diameter and to the calibration of the measuring instrument should be combined to obtain the overall uncertainty in the diameter. The calculation of the combined uncertainty in the diameter is set out in Table A.2.

Table A.2 — Uncertainty budget for mean diameter measurement

Source	Expanded uncertainty U	k	Standard uncertainty u	u^2
Calibration	0.002	2	0.001	0.000 001
Variations			0.000 88	0.000 000 77
Combined	0.002 66[a]	2[a]	0.001 33	0.000 001 77

[a] The calculation of an expanded uncertainty at the 95 % confidence level is not strictly necessary for use in the overall uncertainty evaluation but is included for completeness.

A.3.3 *Uncertainties in the movement per pulse*

The distance moved for a single pulse output is derived by measuring the distance moved for successive counts of 1 000 pulses as the piston is moved from one end of its travel to the other. This process is repeated for each of four passages of the piston. The results obtained are given in Table A.3. The calibration laboratory undertaking the measurements states on the certificate that "the expanded uncertainty (k = 2) for the piston movement is 0.002".

Other methods of establishing the calibration of the length measurement may be adopted depending on how the prover is to be used. Measuring the length at random intervals across the piston's stroke or at random intervals at each end of a full travel may be considered. Similarly a check should be made in case there are cyclic variations in the encoder that could provide uncertainty not seen in a number of fixed 1 000 pulse intervals. Again, for the purpose of this example, the method outlined has been assumed to account adequately for the uncertainties present.

Table A.3 — Measurements of piston movements for successive counts of 1 000 pulses

Passage of piston mm			
1	2	3	4
20.026	20.009	20.001	20.000
20.015	19.993	20.013	19.998
19.993	20.002	20.011	20.007
19.994	19.989	19.998	19.988
20.011	19.994	19.982	20.016
20.002	20.000	20.002	19.995
20.018	19.990	19.987	19.984
20.002	19.994	20.003	19.995
19.987	19.988	20.000	20.012
19.986	20.013	20.002	19.999
19.999	19.994	20.008	19.995
19.997	20.012	19.995	20.001
19.991	19.985	19.989	19.991
20.002	20.000	20.008	19.994
20.007	20.018	19.996	20.007
20.013	19.997	20.000	20.007
20.003	20.007	20.004	20.017
20.000	20.005	20.021	19.993
20.007	20.004	19.981	20.018
19.991	20.015	19.999	20.016

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The mean movement of the piston for a count of 1 000 pulses, $\bar{M}$, is then given by equation (1) as

$$\bar{M} = \frac{M_1 + M_2 + \ldots + M_n}{n}$$
$$= \frac{20.026 + 20.009 + 20.001 + \ldots + 20.015 + 19.999 + 20.016}{80}$$
$$= 20.001 \text{ mm}$$

giving a value for the mean movement per pulse of

$$\bar{m} = \frac{\bar{M}}{1000}$$
$$= \frac{20.001}{1000}$$
$$= 0.020001 \text{ mm}$$

The standard uncertainty of the individual measurements of the movement for a count of 1 000 pulses is given by equation (2) as

$$s_M = \sqrt{\frac{(M_1 - \bar{M})^2 + (M_2 - \bar{M})^2 + \ldots + (M_n - \bar{M})^2}{(n-1)}}$$
$$= \sqrt{\frac{(20.026 - 20.001)^2 + (20.009 - 20.001)^2 + \ldots + (20.016 - 20.001)^2}{(80-1)}}$$
$$= 0.01002 \text{ mm}$$

and the standard uncertainty of the mean movement is then given by equation (5) as

$$u_{\bar{M}} = \frac{s_M}{\sqrt{n}}$$
$$= \frac{0.01002}{\sqrt{80}}$$
$$= 0.00112 \text{ mm}$$

The calculation of the combined uncertainty in the mean movement for 1 000 pulses is set out in Table A.4.

Table A.4 — Uncertainty budget for mean movement for 1 000 pulses

Source	Expanded uncertainty U	k	Standard uncertainty u	u^2
Calibration	0.002	2	0.001	0.000 001
Variations			0.001 12	0.000 001 255
Combined	0.003[a]	2[a]	0.001 50	0.000 002 255

[a] The calculation of an expanded uncertainty at the 95 % confidence level is not strictly necessary for use in the overall uncertainty evaluation but is included for completeness.

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The combined expanded uncertainty for the movement for one pulse is then given by

$$U_{\bar{m}} = \frac{U_{\bar{M}}}{1000}$$
$$= \frac{0.003}{1000}$$
$$= 0.000003 \text{ mm}$$

A.3.4 *Uncertainty in pulse count*

The pulse counter is verified against a reference counter and pulse train and shown to provide an exact count over repeated samples of 20 000 pulses. The uncertainty in the counting of whole pulses is therefore taken to be insignificant. However, when estimating the piston movement in a measured time interval, the pulse counter starts on receipt of the next pulse after the start signal is received from the timer and counting stops on receipt of the last pulse before the stop signal is received. The first pulse counted therefore represents only a part of a pulse and in fact represents any fraction of a pulse with equal probability. The range of possible values is therefore characterized by a rectangular distribution ($k = \sqrt{3}$) with a semi-range of 0.5 pulses. There is a similar uncertainty at the end of the pulse count.

The calculation of the combined uncertainty in the pulse count is set out in Table A.5.

Table A.5 — Uncertainty budget for pulses count

Source	Expanded uncertainty U	k	Standard uncertainty u	u^2
Start	0.5	$\sqrt{3}$	0.289	0.083 3
Finish	0.5	$\sqrt{3}$	0.289	0.083 3
Combined	0.816[a]	2[a]	0.408	0.166 6

[a] The calculation of an expanded uncertainty at the 95 % confidence level is not strictly necessary for use in the overall uncertainty evaluation but is included for completeness.

In calculating the expanded uncertainty of the pulse count, it could be argued that, with the two elements having rectangular distributions of equal size, the combined uncertainty will have a triangular distribution and k should be taken to be $\sqrt{6}$ giving an expanded uncertainty of 1. As the result is only used within the analysis of the overall uncertainty of the prover, it does not matter which approach is used as long as the k factor assumed is noted and used in the next stage to retrieve the correct standard uncertainty.

A.3.5 *Uncertainty in the time measurement*

The time taken is determined by an electronic timer circuit started and stopped from a manual signal. Uncertainties in the timing arise from:

a) the resolution of the timer display; and

b) the calibration of the timer circuit.

As the start–stop signal is also used to gate the pulse count, the operator reaction time plays no part in the timing. Additional uncertainty arises in theory from the differences in response of the timer circuit to the start and stop signals. This source is considered to be negligible.

The timer readout has a resolution of 0.001 s. The resolution is therefore characterized as a rectangular probability distribution having a semi-range of 0.000 5 s.

The calibration certificate for the timer states that “at the 95 % confidence level, the timer is accurate to 0.01 % of the interval timed for intervals between 10 s and 1 000 s”. For a time interval of 20 s, the expanded uncertainty is therefore

$$U_t = \frac{20 \times 0.01}{100} = 0.002 \text{ s}$$

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At the 95 % confidence level, the k factor to obtain the standard uncertainty is 2 and the standard uncertainty is therefore obtained from equation (11) as

$$u_t = \frac{U_t}{k} = \frac{0.002}{2} = 0.001 \text{ s}$$

The calculation of the combined uncertainty in the timing is set out in Table A.6.

Table A.6 — Uncertainty budget for timing

Source	Expanded uncertainty U	k	Standard uncertainty u	u^2
Resolution	0.000 5	$\sqrt{3}$	0.000 289	0.000 000 083 3
Calibration	0.002	2	0.001	0.000 001
Combined	0.002 08[a]	2[a]	0.001 041	0.000 001 083

[a] The calculation of an expanded uncertainty at the 95 % confidence level is not strictly necessary for use in the overall uncertainty evaluation but is included for completeness.

A.4 Sensitivity coefficients

The sensitivity of the flow rate to the measurement of diameter, movement per pulse, pulse count and time can be obtained analytically by the application of equation (17) to the governing equation (A.1).

Thus

$$c_d = \frac{\partial Q}{\partial d} = \frac{\partial}{\partial d}\left(\frac{\pi d^2 mP}{4t}\right) = \frac{2\pi dmP}{4t} = \frac{\pi dmP}{2t}$$

and for the collection of 5 000 pulses in 20 s

$$c_d = \frac{\pi \times 75.00125 \times 0.020001 \times 5000}{2 \times 20}$$
$$= 589.1$$

$$c_m = \frac{\partial Q}{\partial m} = \frac{\partial}{\partial m}\left(\frac{\pi d^2 mP}{4t}\right) = \frac{\pi d^2 P}{4t}$$

and for the collection of 5 000 pulses in 20 s

$$c_m = \frac{\pi \times 75.00125^2 \times 5000}{4 \times 20}$$
$$= 1104500$$

$$c_P = \frac{\partial Q}{\partial P} = \frac{\partial}{\partial P}\left(\frac{\pi d^2 mP}{4t}\right) = \frac{\pi d^2 m}{4t}$$

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and for the collection of 5 000 pulses in 20 s

$$c_P = \frac{\pi \times 75.00125^2 \times 0.020001}{4 \times 20}$$
$$= 4.418$$

$$c_t = \frac{\partial Q}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\pi d^2 mP}{4t}\right) = -\frac{\pi d^2 mP}{4t^2}$$

and for the collection of 5 000 pulses in 20 s

$$c_t = \frac{\pi \times 75.00125^2 \times 0.020001 \times 5000}{4 \times 20^2}$$
$$= -1104.6$$

$$c_t = \frac{\pi \times 75.00125^2 \times 0.020001 \times 5000}{4 \times 20^2}$$
$$= -1104.6$$

A.5 Overall uncertainty in volume flow rate

The calculation of the overall uncertainty in the volume flow rate is set out in Table A.7.

Table A.7 — Uncertainty budget for volume flow rate

	U	*k*	*u*	*c*	*cu*	$(cu)^2$
diameter	0.002 66	2	0.001 33	589.09	0.784 75	0.615 837
move	0.000 003	2	1.502×10^{-6}	1 104 500	1.658 7	2.751 31
pulse	0.816	2	0.408	4.418 2	1.803 7	3.253 46
time	0.002 08	2	0.001 041	–1 104.6	–1.149 7	1.321 72
Combined	5.636 4	2	2.818 2			7.942 33
Nom value	22 091					
% uncertainty	0.026					

A.6 Additional uncertainties

To simplify the example for illustrative purposes, a number of other, potentially large, sources of uncertainty have not been accounted for in this example. The diameter of the cylinder has been established at a reference temperature. The resultant area will change due to temperature and pressure expansion of the tube. Similarly, the radius of the encoder pulley can change with temperature, hence changing the length measurement. As the purpose is to establish the volume flowrate through the meter under test, the effects of temperature and pressure expansion of the fluid between prover and meter should be added to the calculation. These effects carry uncertainties not accounted for in this example. Other sources of uncertainty may well be recognized in a full uncertainty evaluation of the application but are not included here.

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A.7 Uncertainty statement

The evaluation of Table A.7 applies only to the case when the minimum pulse count of 5 000 pulses is collected in the minimum time of 20 s. The volume collected is then

$$V = \frac{\pi d^2 mP}{4}$$
$$= \frac{\pi \times 75.00125^2 \times 0.020001 \times 5000}{4}$$
$$= 4418233 \ \text{mm}^3$$
$$= 441.8233 \ \text{ml}$$

and the flow rate is

$$Q = \frac{V}{t}$$
$$= \frac{441.8233}{20}$$
$$= 22.0912 \ \text{ml/s}$$

The uncertainty of the flowrate can be expressed as follows:

When a volume of 440 ml is collected in 20 s, the expanded uncertainty (k = 2) in the flow rate is 6 mm^3/s, or 0.026 %.

As resolution uncertainties will have different effects on the final uncertainty if larger volumes are collected or if the flow rate is slower and the collection time is longer, the analysis should ideally be repeated at the corners of the operating envelope and the final uncertainty statement should either quote the worst case uncertainty or specify individual uncertainties for various parts of the operating envelope.

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Annex B (normative)
Setting out the calculation

B.1 Spreadsheet calculation of sensitivity coefficients

Table B.1 shows a spreadsheet set out to calculate the sensitivity coefficients for a measurement result, y, based on N input quantities, X_1, X_2,…,X_N. The second row identifies the contents of each column and the third row shows the nominal values of each input quantity leading to the calculation in column 7 of the nominal value of the output estimate y. In subsequent rows column 2 contains the chosen increment for the calculation of the sensitivity coefficient. The rows are paired with the chosen increment added to the nominal value of the appropriate input in columns 3 to 6 in the first row and subtracted from it in the second. The result in column 7 is then the value of the output y resulting from the incremented input quantity. Columns 8 and 9 then contain the calculation of the absolute and relative sensitivity coefficients respectively. The table can be extended to any number of input parameters with each having its own row, its own increment in column 2 of that row and its own column of nominal values by inserting columns in the range 3 to 6. Problems with calculation round-off or non-linear responses resulting in erroneous values of c can easily be checked by altering the increment in column 2 and observing any change in the sensitivity coefficients in columns 8 and 9.

Table B.1 — Spreadsheet set out for the calculation of sensitivity coefficients

1	2	3	4	5	6	7	8	9
Sensitivity coefficient	Increment	X_1	X_2	…	X_N	Y	c	c^*
	—	x_1	x_2	…	x_N	$y_{\text{nom}} = f(x_1, x_2, \ldots, x_i, \ldots, x_N)$		
c_1	$\Delta x_1 \approx u(x_1)$	$x_1 + \Delta x_1$	x_2	…	x_N	$y_{1+} = f(x_1 + \Delta x_1, x_2, \ldots, x_i, \ldots, x_N)$	$\dfrac{(y_{1+} - y_{1-})}{2\Delta x_1}$	$\dfrac{c_1 x_1}{y_{\text{nom}}}$
c_1	$\Delta x_1 \approx u(x_1)$	$x_1 - \Delta x_1$	x_2	…	x_N	$y_{1-} = f(x_1 - \Delta x_1, x_2, \ldots, x_i, \ldots, x_N)$		
c_2	$\Delta x_2 \approx u(x_2)$	x_1	$x_2 + \Delta x_2$	…	x_N	$y_{2+} = f(x_1, x_2 + \Delta x_2, \ldots, x_i, \ldots, x_N)$	$\dfrac{(y_{2+} - y_{2-})}{2\Delta x_2}$	$\dfrac{c_2 x_2}{y_{\text{nom}}}$
c_2	$\Delta x_2 \approx u(x_2)$	x_1	$x_2 - \Delta x_2$	…	x_N	$y_{2-} = f(x_1, x_2 - \Delta x_2, \ldots, x_i, \ldots, x_N)$		

B.2 The uncertainty budget table

The simple step-by-step approach outlined in Clauses **5** to **10** allows the straightforward evaluation of the uncertainty of a measurement quantity associated with a complex functional relationship involving several input quantities. However, there may be many factors to be taken into account and care is needed to ensure that they have been properly accounted for. By using a computer spreadsheet, the entire process can be presented in a clear and logical progression. The approach recommended is commonly implemented in what is known as an uncertainty budget table and an example is shown in Table B.2. This approach will be seen later to have advantages when seeking to identify the major sources of uncertainty or to analyse the benefits of changes in parts of the system.

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Table B.2 — Sample layout of uncertainty budget table

1	2	3	4	5	6	7	8	9	10	11	12
Input quantity	**Nominal value** x_i	**Expanded uncertainty or limit relative**	**Expanded uncertainty or limit absolute**	**Probability distribution**	**Divisor** k	**Standard uncertainty** u_{xi} or u^*_{xi}	**Sensitivity coefficient** c_{xi} or c^*_{xi}	**Contribution to uncertainty** $u_{xi} \cdot c_{xi}$	**Correlated group contribution**	**Contribution squared** $(u_{xi} \cdot c_{xi})^2$	**Rank**
e.g. pressure 1 calibration	105 000	1.0 %	1 050	Normal	2	525	–1	–525	25	625	3
e.g. pressure 2 calibration	110 000	1.0 %	1 100	Normal	2	550	1	550			
e.g. pressure 1 resolution	105 000		98	Rectangular	$\sqrt{3}$	56.4	–1	–56.4		3 179	1=
e.g. pressure 2 resolution	110 000		98	Rectangular	$\sqrt{3}$	56.4	1	56.4		3 179	1=
Calculated result	5 000	$U^*_c(y)$ = 3.34 %	$U_c(y)$ = 167	←	2	$u_c(y)$ = 83.56	←	=$\sqrt{}$6 983	←	6 983	

NOTE 1 Table is set out for the example of a differential pressure obtained by reading each pressure with the same transducer.

NOTE 2 Functional relationship:

$\Delta p = p_2 - p_1$

NOTE 3 p_2 = 110 000 Pa, p_1 = 105 000 Pa.

NOTE 4 Calibration uncertainty 1 % of reading given as an expanded uncertainty at 95 % confidence level.

NOTE 5 Resolution uncertainty obtained from a-to-d converter resolution limit of 1 part in 1 024 with full scale = 2 bar = 200 000 Pa, ∴ half range = 200 000/1 024/2 = 98 Pa.

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The columns within the budget table are as follows:

Column 1: Source of uncertainty = *input*

Each source is assigned a separate row within the table and where several sources contribute to uncertainty in one input quantity these are grouped together.

Column 2: Nominal value of the input parameter = *input*

This column is included for use when some input uncertainties are available as percentages and some in absolute terms.

Column 3: Input expanded uncertainty relative = *input (optional)*

Column 4: Input expanded uncertainty absolute = *input or column 3/column 2*

In a practical evaluation some uncertainties might be available as absolute values and others as relative values. The inclusion of the nominal value in column 2 permits all uncertainties to be brought to a common basis for use in the analysis.

Column 5: Probability distribution

This column is used to record the distribution assumed for each source. Where the distribution to use is unclear, normal practice is to use a rectangular distribution.

Column 6: Divisor

Each probability distribution has its own divisor to obtain the standard uncertainty from the expanded value or limits (see **8.3**).

Column 7: Standard uncertainty = *column 3/column 6 (relative) or column 4/column 6 (absolute)*

At this point, the decision should be made whether to use absolute or relative terms. The problem of arbitrary zeros (examples 5 and 6) can be avoided by using absolute terms throughout the analysis and converting to a relative value for the final answer, if required. As was highlighted in **10.2** and **10.3**, all rows should be in the same terms.

Column 8: Sensitivity coefficients

It is vitally important to be consistent. If column 7 is absolute, so should column 8 be; if column 7 is relative, so should column 8 be; and again all rows should be in the same terms.

Column 9: Contribution to uncertainty = *column 7 × column 8*

This is the product of standard uncertainty and sensitivity coefficient.

Column 10: Correlated group uncertainty contribution

Where sources of uncertainty are correlated, perhaps through the use of the same instrument to take measurements, their contributions from column 9 can be added to provide a single entry in column 10.

Column 11: Contribution squared = $(column\ 9)^2$

Entries in this column are the squares of the entries in column 9. However, when entries in column 9 have been summed arithmetically in a correlated group in column 10, it is the column 10 entry that is squared and carried forward into column 11 and the relevant column 9 entries do not contribute directly to column 11.

The summation of all sources is then a matter of summing all entries in column 11, and the calculation proceeds back along the bottom row of the table from right to left. The standard uncertainty of the output, $u(y)$, is then the square root of the column 11 summation. The divisor (column 6) becomes the multiplier appropriate to the confidence level required for the final uncertainty and the expanded uncertainty of the result [$U(y) = ku(y)$] appears at the foot of column 3 or 4 as appropriate.

Uncertainty evaluation often calls for judgement in assigning limits or distributions to an input uncertainty. When the layout of Table B.2 is used the effect of such decisions can be checked by changing the appropriate cell and observing the change in the final answer.

Similarly, when the extent of any correlation is unclear, the result can be recalculated by using or not using column 10 and comparing the final answers.

An uncertainty evaluation is often carried out to identify the aspects of a measurement system that need to be addressed to improve the measurement quality. The layout of Table B.2 is ideal for this purpose since columns 9 and 11 identify the principal contributions to the overall uncertainty. An extra column (column 12) can be added to show the ranking of the elements in column 9 or 11.

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Bibliography

ISO/TS 21748, *Guide to the use of repeatability, reproducibility and trueness estimates in measurement uncertainty estimation.*

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