BS EN 61710:2013

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Power law model — Goodness-of-fit tests and estimation methods

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National foreword

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The UK participation in its preparation was entrusted to Technical Committee DS/1, Dependability.

A list of organizations represented on this committee can be obtained on request to its secretary.

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Power law model - Goodness-of-fit tests and estimation methods
(IEC 61710:2013)

Modèle de loi en puissance -
Essais d'adéquation et méthodes d'estimation des paramètres
(CEI 61710:2013)

Potenzgesetz-Modell -
Anpassungstests und Schätzverfahren
(IEC 61710:2013)

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Foreword

The text of document 56/1500/FDIS, future edition 2 of IEC 61710, prepared by IEC/TC 56 "Dependability" was submitted to the IEC-CENELEC parallel vote and approved by CENELEC as EN 61710:2013.

The following dates are fixed:

• latest date by which the document has to be implemented at national level by publication of an identical national standard or by endorsement	(dop)	2014-03-26
• latest date by which the national standards conflicting with the document have to be withdrawn	(dow)	2016-06-26

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The text of the International Standard IEC 61710:2013 was approved by CENELEC as a European Standard without any modification.

In the official version, for Bibliography, the following notes have to be added for the standards indicated:

IEC 61703	NOTE	Harmonised as EN 61703.
IEC 61164:2004	NOTE	Harmonised as EN 61164:2004 (not modified).

Annex ZA
(normative)

Normative references to international publications with their corresponding European publications

The following documents, in whole or in part, are normatively referenced in this document and are indispensable for its application. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

NOTE When an international publication has been modified by common modifications, indicated by (mod), the relevant EN/HD applies.

Publication	Year	Title	EN/HD	Year
IEC 60050-191	1990	International Electrotechnical Vocabulary (IEV) - Chapter 191: Dependability and quality of service	-	-

CONTENTS

INTRODUCTION

This International Standard describes the power law model and gives step-by-step directions for its use. There are various models for describing the reliability of repairable items, the power law model being one of the most widely used. This standard provides procedures to estimate the parameters of the power law model and to test the goodness-of-fit of the power law model to data, to provide confidence intervals for the failure intensity and prediction intervals for the length of time to future failures. An input is required consisting of a data set of times at which relevant failures occurred, or were observed, for a repairable item or a set of copies of the same item, and the time at which observation of the item was terminated, if different from the time of final failure. All output results correspond to the item type under consideration.

Some of the procedures can require computer programs, but these are not unduly complex. This standard presents algorithms from which computer programs should be easy to construct.

POWER LAW MODEL – GOODNESS-OF-FIT TESTS AND ESTIMATION METHODS

1 Scope

This International Standard specifies procedures to estimate the parameters of the power law model, to provide confidence intervals for the failure intensity, to provide prediction intervals for the times to future failures, and to test the goodness-of-fit of the power law model to data from repairable items. It is assumed that the time to failure data have been collected from an item, or some identical items operating under the same conditions (e.g. environment and load).

2 Normative references

The following documents, in whole or in part, are normatively referenced in this document and are indispensable for its application. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

IEC 60050-191:1990, *International Electrotechnical Vocabulary (IEV) – Chapter 191: Dependability and quality of service*

3 Terms and definitions

For the purposes of this document, the terms and definitions of IEC 60050-191 apply.

4 Symbols and abbreviations

The following symbols and abbreviations apply:

β	shape parameter of the power law model
$\hat{\beta}$	estimated shape parameter of the power law model
β_{LB}, β_{UB}	lower, upper confidence limits for β
C^2	Cramer-von-Mises goodness-of-fit test statistic
$C^2_{1-\gamma}(M)$	critical value for the Cramer-von-Mises goodness-of-fit test statistic at γ level of significance
χ^2	Chi-square goodness-of-fit test statistic
$\chi^2_{\gamma}(\upsilon)$	γth fractile of the χ^2 distribution with υ degrees of freedom
d	number of intervals for groups of failures
$E[N(t)]$	expected accumulated number of failures up to time t
$E[t_j]$	expected accumulated time to jth failure

$\hat{E}[N[t(i)]]$	estimated expected accumulated number of failures up to $t(i)$
$\hat{E}[t_j]$	estimated expected accumulated time to jth failure
$F_\gamma(\nu_1,\nu_2)$	γ th fractile for the F distribution with (ν_1,ν_2) degrees of freedom
i	general purpose indicator
j	general purpose indicator
k	number of items
L, U	multipliers used in calculation of confidence intervals for failure intensity
λ	scale parameter of the power law model
$\hat{\lambda}$	estimated scale parameter of the power law model
M	parameter for Cramer-von-Mises statistical test
N	number of relevant failures
N_j	number of failures for jth item
$N(t)$	accumulated number of failures up to time t
$N[t(i)]$	accumulated number of failures up to time $t(i)$
R	difference between the order number of future (predicted) failure and order number of last (observed) failure
T	accumulated relevant time
T^*	total accumulated relevant time for time terminated test
T_j	total accumulated relevant time for jth item
T_{RL}, T_{RU}	lower, upper prediction limits for the length of time to the Rth future failure
$\hat{T}_{N+1}$	estimated median time to $(N+1)$th failure
t_i	accumulated relevant time to the ith failure
t_{ij}	ith failure time for jth item
t_N	total accumulated relevant time for failure terminated test
t_{Nj}	total accumulated relevant time to Nth failure of jth item
$t(i-1), t(i)$	endpoints of ith interval of time for grouped failures
$z(t)$	failure intensity at time t
$\hat{z}(t)$	estimated failure intensity at time t
z_{LB}, z_{UB}	lower, upper confidence limits for failure intensity

5 Power law model

The statistical procedures for the power law model use the relevant failure and time data from the test or field studies. The basic equations for the power law model are given in this clause. Background information on the model is given in Annex A and examples of its application are given in Annex B.

The expected accumulated number of failures up to test time t is given by:

$$E[N(t)] = \lambda t^{\beta} \text{ with } \lambda > 0, \beta > 0, t > 0$$

where

λ is the scale parameter;

β is the shape parameter ($0 < \beta < 1$ corresponds to a decreasing failure intensity; $\beta = 1$ corresponds to a constant failure intensity; $\beta > 1$ corresponds to an increasing failure intensity).

The failure intensity at time t is given by:

$$z(t) = \frac{d}{dt} E[N(t)] = \lambda \beta t^{\beta - 1} \text{ with } t > 0$$

Thus the parameters λ and β both affect the failure intensity in a given time.

Methods are given in 7.2 for maximum likelihood estimation of the parameters of λ and β. Subclause 7.3 gives goodness-of-fit tests for the model and 7.4 and 7.5 give confidence interval procedures. Subclause 7.6 gives prediction interval procedures and 7.7 gives tests for the equality of the shape parameters. The model is simple to evaluate. However when $\beta < 1$, theoretically $z(0) = \infty$ (i.e. $z(t)$ tends to infinity as t tends to zero) and $z(\infty) = 0$ (i.e. $z(t)$ tends to zero as t tends to infinity); but this theoretical limitation does not generally affect its practical use.

6 Data requirements

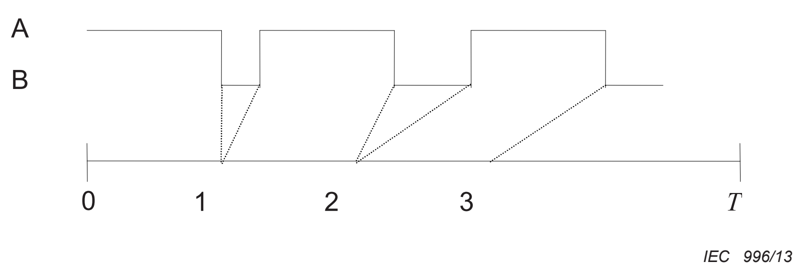
6.1 General

6.1.1 Case 1 – Time data for every relevant failure for one or more copies from the same population

The normal evaluation methods assume the observed times to be exact times of failure of a single repairable item or a set of copies of the same repairable item. The figures below illustrate how the failure times are calculated for three general cases.

6.1.2 Case 1a) – One repairable item

For one repairable item observed from time 0 to time T, the relevant failure time, t_i, is the elapsed operating time (that is, excluding repair and other down times) until the occurrence of the i-th failure as shown in Figure 1.



Key

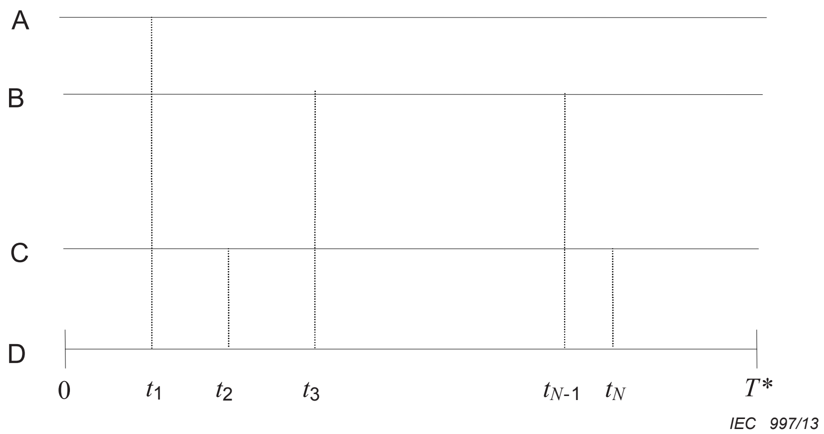
A operating time, B down time

Figure 1 – One repairable item

Time terminated data are observed to T^*, which is not a failure time, and failure terminated data are observed to t_N, which is the time of the Nth failure. Time terminated and failure terminated data use slightly different formulae.

6.1.3 Case 1b) – Multiple items of the same kind of repairable item observed for the same length of time

It is assumed there are k items, which all represent the same population. That is, they are nominally identical items operating under the same conditions (e.g. environment and load). When all items are observed to time T^*, which is not a failure time (i.e. time terminated data), then the failure time data are combined by superimposing failure times $(t_i, i = 1, 2..., N)$ for all k items on the same time line as shown in Figure 2.



Key

A item 1

B item 2

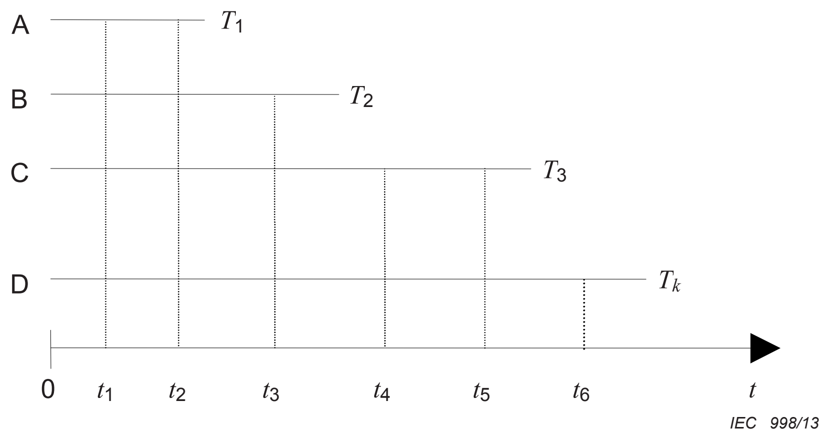
C item k

D superimposed process

Figure 2 – Multiple items of the same kind of repairable item observed for same length of time

6.1.4 Case 1c) – Multiple repairable items of the same kind observed for different lengths of time

When all items do not operate for the same period of time, then the time at which observation of the jth item is terminated $T_j (j = 1, 2, ..., k)$, where $T_1 < T_2 < ... < T_k$, is noted. The failure data are combined by superimposing all the failure times for all k items on the same time line as shown in Figure 3. The times to failure are $t_i, i = 1, 2, ..., N$, where N = the total number of failures observed accumulated over the k items.



Key

A item 1

B item 2

C item 3

D item k

t time

Figure 3 – Multiple repairable items of the same kind observed for different lengths of time

If each item is a software system then the repair action should be done to the other systems which did not fail at that time.

6.2 Case 2 – Time data for groups of relevant failures for one or more repairable items from the same population

This alternative method is used when there is at least one copy of an item and the data consist of known time intervals, each containing a known number of failures.

The observation period is over the interval $(0, T)$ and is partitioned into d intervals at times $0 < t(1) < t(2) < ... < t(d)$. The ith interval is the time period between $t(i-1)$ and $t(i)$, where $i = 1, 2,..., d$, $t(0) = 0$ and $t(d) = T$. It is important to note that the interval lengths and the number of failures per interval need not be the same.

6.3 Case 3 – Time data for every relevant failure for more than one repairable item from different populations

It is assumed there are k items which do not represent the same population and are to be compared. It should be noted that if each item is to be considered individually then it is appropriate to use case 1a) in 6.1.2.

If direct comparisons of the items are to be made then as an extension of 6.1 the following notation is used:

t_{ij} denotes the ith failure time for the process corresponding to the jth item;

N_j denotes the number of failures observed for the jth item;

t_{N_j} is the time of the Nth failure for the jth item;

where i = 0, 1, 2, … N_j and j = 1, 2, …k.

7 Statistical estimation and test procedures

7.1 Overview

In case 1 – time data for every relevant failure – the formulae given for failure terminated data assume one repairable item, that is $k=1$. All output results correspond to that item. The formulae given for time terminated data assume k copies of the item observed for the same length of time. If there is only one repairable item then $k=1$. The point estimation procedures for all the aforementioned cases are given in 7.2.1. The appropriate procedures for the case when all copies are observed for different lengths of time are given in 7.2.2. Procedures for the case of time data for groups of relevant failures are given in 7.2.3.

An appropriate goodness-of-fit test, as described in 7.3 shall be performed after the parameter estimation procedures of 7.2. Note that these tests, and the procedures given in 7.4 to 7.7 for constructing interval estimates and carrying out statistical tests, distinguish only between the cases of time data for every relevant failure (i.e. all instances of case 1 data – 1a), 1b) and 1c)) and time data for groups of relevant failures (i.e. case 2)).

The inference procedures that follow provide approximate estimates in some circumstances and so caution is required if they are to be applied if the number of observed failures is less than 10.

7.2 Point estimation

7.2.1 Case 1a) and 1b) – Time data for every relevant failure

This method applies only when the time of failure has been logged for every failure as described in 6.1.2 and 6.1.3.

Step 1: Calculate the summation:

$$S_1 = \sum_{j=1}^{N} \ln\left(\frac{T^*}{t_j}\right) \qquad \textit{(time terminated)}$$

$$S_2 = \sum_{j=1}^{N} \ln\left(\frac{t_N}{t_j}\right) \qquad \textit{(failure terminated)}$$

Step 2: Calculate the (unbiased) estimate of the shape parameter β from the formula:

$$\hat{\beta} = \frac{N-1}{S_1} \qquad \textit{(time terminated)}$$

$$\hat{\beta} = \frac{N-2}{S_2} \qquad \textit{(failure terminated)}$$

Step 3: Calculate the estimate of the scale parameter λ from the formula:

$$\hat{\lambda} = \frac{N}{k\left(T^*\right)^{\hat{\beta}}} \qquad \textit{(time terminated)}$$

$$\hat{\lambda} = \frac{N}{k\left(t_N\right)^{\hat{\beta}}} \qquad \text{(failure terminated)}$$

Step 4: Calculate the estimate of the failure intensity $z(t)$, for any time $t > 0$, from the formula:

$$\hat{z}(t) = \hat{\lambda}\hat{\beta}t^{\hat{\beta}-1}$$

$\hat{z}(t)$ estimates the current failure intensity for t over the range represented by the data. "Extrapolated" estimates for a future t may be obtained similarly, but should be used with the usual caution associated with extrapolation.

Step 5: Given N observed failures the last of which occurred at t_N, the median time to the $(N+1)$th failure can be estimated from the formula:

$$\hat{T}_{N+1} = t_N \exp\left[\frac{0{,}5^{\frac{-1}{N+1}} - 1}{N\hat{\beta}/(N-1)}\right] \qquad \text{(time terminated)}$$

$$\hat{T}_{N+1} = t_N \exp\left[\frac{0{,}5^{\frac{-1}{N+1}} - 1}{N\hat{\beta}/(N-2)}\right] \qquad \text{(failure terminated)}$$

7.2.2 Case 1c) – Time data for every relevant failure

This method applies only when the time of failure has been logged for every failure as described in 6.1.4.

Step 1: Assemble the data into the times to failure, t_i, $i=1,2,..,N,$ where N is the total number of failures over the k copies and T_j, $j=1,2,...,k$, is the end of the observation period for the jth copy.

Step 2: The maximum likelihood estimate of the shape parameter β is the value of $\hat{\beta}$ which satisfies the formula:

$$\frac{N}{\hat{\beta}} + \sum_{i=1}^{N} \ln t_i - \frac{N\sum_{j=1}^{k} T_j^{\hat{\beta}} \ln T_j}{\sum_{j=1}^{k} T_j^{\hat{\beta}}} = 0$$

An iterative method shall be used to solve the formula for $\hat{\beta}$.

Step 3: Calculate the estimate of the scale parameter λ from the formula:

$$\hat{\lambda} = \frac{N}{\sum_{j=1}^{k} T_j^{\hat{\beta}}}$$

Step 4: Calculate the estimate of the failure intensity $z(t)$, for any time $t > 0$, from the formula:

$$\hat{z}(t) = \hat{\lambda}\,\hat{\beta}\, t^{\hat{\beta}-1}$$

$\hat{z}(t)$ estimates the current failure intensity for t over the range represented by the data. "Extrapolated" estimates for a future t may be obtained similarly, but should be used with the usual caution associated with extrapolation.

7.2.3 Case 2 – Time data for groups of relevant failures

This method applies when the data set consists of known time intervals, each containing a known number of failures as described in 6.2.

Step 1: Assemble into a data set the number of relevant failures N_i recorded in the ith interval $[t(i-1), t(i)], i = 1, 2, ..., d$. The total number of relevant failures is

$$N = \sum_{i=1}^{d} N_i$$

Step 2: The maximum likelihood estimate of the shape parameter β is the value of $\hat{\beta}$ which satisfies the formula:

$$\sum_{i=1}^{d} N_i \left[\frac{[t(i)]^{\hat{\beta}} \ln t(i) - [t(i-1)]^{\hat{\beta}} \ln t(i-1)}{[t(i)]^{\hat{\beta}} - [t(i-1)]^{\hat{\beta}}} - \ln t(d) \right] = 0$$

Note that $[t(0)]^{\hat{\beta}} = 0$ and $[t(0)]^{\hat{\beta}} \ln t(0) = 0$. All $t(.)$ terms may be normalized with respect to $t(d)$ and then the final term $\ln\left[t(d)\right]$ disappears. An iterative method shall be used to solve the formula for $\hat{\beta}$.

Step 3: Calculate the estimate of the scale parameter λ from the formula:

$$\hat{\lambda} = \frac{N}{t(d)^{\hat{\beta}}}$$

Step 4: Calculate the estimate of the failure intensity $z(t)$, for any test time $t > 0$, from the formula:

$$\hat{z}(t) = \hat{\lambda}\,\hat{\beta}\, t^{\hat{\beta}-1}$$

$\hat{z}(t)$ estimates the current failure intensity for t over the range represented by the data. "Extrapolated" estimates for a future t may be obtained similarly, but should be used with the usual caution associated with extrapolation.

7.3 Goodness-of-fit tests

7.3.1 Case 1 – Time data for every relevant failure

7.3.1.1 Cramer-von-Mises test

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 or step 2 in 7.2.2.

Step 2: Calculate the Cramer-von-Mises goodness-of-fit test statistic given by the formula:

$$C^2 = \frac{1}{12M} + \sum_{j=1}^{M}\left[\left(\frac{t_j}{T}\right)^{\hat{\beta}} - \left(\frac{2j-1}{2M}\right)\right]^2$$

where

$M = N$ and $T = T^*$ (*time terminated*)

$M = N - 1$ and $T = t_N$ (*failure terminated*)

Step 3: Select the critical value $C^2_{0,90}(M)$ for the Cramer-von-Mises test corresponding to M from Table 1, which gives critical values at a 10 % significance level.

Step 4: If:

$$C^2 > C^2_{0,90}(M)$$

then the hypothesis that the power law model fits the data cannot be accepted. Otherwise, on the basis of the data analysed, the power law model can be used as a working hypothesis.

7.3.1.2 Graphical procedure

When the failure times are known, the graphical procedure described below may be used to obtain additional information about the correspondence between the model and the data. This involves plotting the expected time to the jth failure, $E(t_j)$, against the observed time to the jth failure. Further details about the approach are given in Annexes A and B.

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 and $\hat{\lambda}$ from step 3 in 7.2.1.

Step 2: Calculate the estimate of the expected time to the jth failure, $j=1,2,..,N$, from the formula:

$$\hat{E}(t_j) = \left(\frac{j}{\hat{\lambda}}\right)^{\frac{1}{\hat{\beta}}}$$

Step 3: Plot $\hat{E}(t_j)$ against t_j on identical linear scales. The visual agreement of these points with a line of 45 ° through the origin is a subjective measure of the applicability of the model.

7.3.2 Case 2 – Time data for groups of relevant failures

7.3.2.1 Chi-square test

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.3 and $\hat{\lambda}$ from step 3 in 7.2.3.

Step 2: Calculate the expected number of failures in the time interval $[t(i-1), t(i)]$ which is approximated by:

$$e_i = \hat{\lambda}\left\{[t(i)]^{\hat{\beta}} - [t(i-1)]^{\hat{\beta}}\right\}$$

Step 3: For each interval, e_i shall not be less than 5, and if necessary, adjacent intervals should be combined before the test. For d intervals (after combination if necessary) and with N_i the same as in 7.2.3, calculate the statistic:

$$\chi^2 = \sum_{i=1}^{d} \frac{(N_i - e_i)^2}{e_i}$$

Step 4: Select the critical value from a χ^2 distribution with $(d-2)$ degrees of freedom and a 10 % significance level from Table 2, i.e. $\chi^2_{0,90}(d-2)$.

Step 5: If the test statistic χ^2 exceeds the critical value $\chi^2_{0,90}(d-2)$ then the hypothesis that the power law model fits the data cannot be accepted. Otherwise, on the basis of the data analysed, the power law model can be used as a working hypothesis.

The Chi-square test is a large sample test and so will need large data sets to detect deviations from the power law model that are practically important.

7.3.2.2 Graphical procedure

When the data set consists of known time intervals, each containing a known number of failures, the graphical procedure described below may be used to obtain additional information about the correspondence between the model and the data. This involves plotting the expected number of failures against those observed at each endpoint. Further details of the approach are given in B.5.

Step 1: For each endpoint $t(i)$, calculate the observed number of failures from 0 to $t(i)$ from the formula:

$$N[t(i)] = \sum_{j=1}^{i} N_j$$

Step 2: Calculate the estimate of the corresponding expected number of failures $E[N[t(i)]]$ from the formula:

$$\hat{E}[N[t(i)]] = \hat{\lambda} t(i)^{\hat{\beta}}$$

Step 3: Plot $\hat{E}[N[t(i)]]$ against $N[t(i)]$on identical linear scales. The visual agreement of these points with a line of 45 ° through the origin is a subjective measure of the applicability of the model.

7.4 Confidence intervals for the shape parameter

7.4.1 Case 1 – Time data for every relevant failure

The shape parameter β in the power law model determines if the failure intensity changes with time. If $0 < \beta < 1$, there is decreasing failure intensity; if $\beta = 1$, there is a constant failure intensity; if $\beta > 1$, there is an increasing failure intensity.

For a two-sided confidence interval for β when individual failure times are available, follow the steps below as appropriate for time and failure terminated data.

Two-sided 90 % confidence interval for β – Time terminated data

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 or from step 2 in 7.2.2.

Step 2: Calculate:

$$D_L = \frac{\chi^2_{0,05}(2N)}{2(N-1)}$$

$$D_U = \frac{\chi^2_{0,95}(2N)}{2(N-1)}$$

where the fractiles of the χ^2 distribution are given in Table 2.

Step 3: Calculate the lower confidence limit for β from the formula:

$$\beta_{LB} = D_L \hat{\beta}$$

and the upper confidence limit for β from the formula:

$$\beta_{UB} = D_U \hat{\beta}$$

Step 4: The two-sided 90 % confidence interval for β is given by (β_{LB}, β_{UB}).

NOTE One-sided 95 % lower and upper limits for β are β_{LB} and β_{UB}, respectively.

Two-sided 90 % confidence interval for β – Failure terminated data

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1.

Step 2: Calculate:

$$D_L = \frac{\chi^2_{0,05}(2(N-1))}{2(N-2)}$$

$$D_U = \frac{\chi^2_{0,95}(2(N-1))}{2(N-2)}$$

where the fractiles of the χ^2 distribution are given in Table 2.

Step 3: Calculate the lower confidence limit for β from the formula:

$$\beta_{LB} = D_L \hat{\beta}$$

and the upper confidence limit for β from the formula:

$$\beta_{UB} = D_U \hat{\beta}$$

Step 4: The two-sided 90 % confidence interval for β is given by (β_{LB}, β_{UB}).

NOTE One-sided 95 % lower and upper limits for β are β_{LB} and β_{UB}, respectively.

7.4.2 Case 2 – Time data for groups of relevant failures

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.3.

Step 2: Calculate:

$$P(i) = \frac{t(i)}{t(d)} \qquad \text{with} \qquad i = 1, 2, ..., d$$

Step 3: Calculate the expression:

$$A = \sum_{i=1}^{d} \frac{\left[[P(i)]^{\hat{\beta}} \ln[P(i)]^{\hat{\beta}} - [P(i-1)]^{\hat{\beta}} \ln[P(i-1)]^{\hat{\beta}} \right]^2}{\left[[P(i)]^{\hat{\beta}} - [P(i-1)]^{\hat{\beta}} \right]}$$

Step 4: Calculate:

$$C = \frac{1}{\sqrt{A}}$$

Step 5: For an approximate two-sided 90 % confidence interval for β, calculate:

$$S = \frac{1{,}64C}{\sqrt{N}}$$

where N is the total number of failures.

Step 6: Calculate the lower confidence limit for β from the formula:

$$\beta_{LB} = \hat{\beta}(1-S)$$

and the upper confidence limit for β from the formula:

$$\beta_{UB} = \hat{\beta}(1+S)$$

Step 7: The two-sided 90 % confidence interval for β is given by (β_{LB}, β_{UB}).

NOTE One-sided 95 % lower and upper limits for β are β_{LB} and β_{UB}, respectively.

7.5 Confidence intervals for the failure intensity

7.5.1 Case 1 – Time data for every relevant failure

Step 1: Calculate $\hat{z}(t)$ from step 4 in 7.2.1 or step 4 in 7.2.2.

Step 2: For a two-sided 90 % confidence interval refer to Table 3 (*time terminated*) and Table 4 (*failure terminated*) and locate values of L and U for the appropriate sample size N.

Step 3: Calculate the lower confidence limit for $z(t)$ from the formula*:*

$$z_{LB} = \frac{\hat{z}(t)}{U}$$

and the upper confidence limit for $z(t)$ from the formula:

$$z_{UB} = \frac{\hat{z}(t)}{L}$$

Step 4: The two-sided 90 % confidence interval for $z(t)$ is given by (z_{LB}, z_{UB}).

NOTE One-sided 95 % lower and upper limits for $z(t)$ are z_{LB} and z_{UB}, respectively.

7.5.2 Case 2 – Time data for groups of relevant failures

Step 1: Calculate $\hat{z}(t)$ from step 4 in 7.2.3.

Step 2: Calculate:

$$P(i) = \frac{t(i)}{t(d)} \qquad \text{with} \qquad i = 1, 2, \ldots, d$$

Step 3: Calculate:

$$A = \sum_{i=1}^{d} \frac{\left[[P(i)]^{\hat{\beta}} \ln[P(i)]^{\hat{\beta}} - [P(i-1)]^{\hat{\beta}} \ln[P(i-1)]^{\hat{\beta}} \right]^2}{\left[[P(i)]^{\hat{\beta}} - [P(i-1)]^{\hat{\beta}} \right]}$$

Step 4: Calculate:

$$D = \sqrt{\frac{1}{A} + 1}$$

Step 5: For an approximate two-sided 90 % confidence interval for $z(t)$ calculate:

$$S = \frac{1{,}64D}{\sqrt{N}}$$

where N is the cumulative number of relevant failures.

Step 6: The lower confidence limit on $z(t)$ is given by:

$$z_{LB} = \frac{\hat{z}(t)}{1+S}$$

and the upper confidence limit on $z(t)$ is given by:

$$z_{UB} = \frac{\hat{z}(t)}{1-S}$$

Step 7: The two-sided 90 % confidence interval for $z(t)$ is given by (z_{LB}, z_{UB}).

NOTE One-sided 95 % lower and upper limits for $z(t)$ are z_{LB} and z_{UB}, respectively.

7.6 Prediction intervals for the length of time to future failures of a single item

7.6.1 Prediction interval for length of time to next failure for case 1 – Time data for every relevant failure

For a two-sided 90 % prediction interval for the time to the $(N+1)$th failure T_{N+1}, that is, the next future failure given that N failures have occurred at times $t_1, t_2, \ldots, t_N$, follow the steps below as appropriate for time and failure terminated data.

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 or from step 2 in 7.2.2.

Step 2: Calculate the lower prediction limit for T_{N+1} from the formula:

$$T_{1L} = t_N \exp\left[\frac{0{,}95^{\frac{-1}{N-1}} - 1}{N\hat{\beta}/(N-1)}\right] \qquad \textit{(time terminated)}$$

$$T_{1L} = t_N \exp\left[\frac{0{,}95^{\frac{-1}{N-1}} - 1}{N\hat{\beta}/(N-2)}\right] \qquad \textit{(failure terminated)}$$

and the upper prediction limit for T_{N+1} from the formula:

$$T_{1U} = t_N \exp\left[\frac{0{,}05^{\frac{-1}{N-1}} - 1}{N\hat{\beta}/(N-1)}\right] \qquad \textit{(time terminated)}$$

$$T_{1U} = t_N \exp\left[\frac{0{,}05^{\frac{-1}{N-1}} - 1}{N\hat{\beta}/(N-2)}\right] \qquad \textit{(failure terminated)}$$

Step 3: The two-sided 90 % prediction interval for T_{N+1} is given by (T_{1L}, T_{1U}).

NOTE One-sided 95 % lower and upper limits for T_{N+1} are T_{1L} and T_{1U}, respectively.

7.6.2 Prediction interval for length of time to *R*th future failure for case 1 – Time data for every relevant failure

For an approximate two-sided 90 % prediction interval for the time to the $(N+R)$th failure T_{N+R}, that is, the Rth future failure given that N failures have occurred at times $t_1, t_2, \ldots, t_N$, follow the steps below as appropriate for failure and time terminated data.

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 or from step 2 in 7.2.2.

Step 2: Calculate:

$$G = \left[\frac{(N-0{,}5)(N+R-0{,}5)}{NR}\right] \ln\left[\frac{N+R-0{,}5}{N-0{,}5}\right]$$

Step 3: Calculate:

$$V = 2NG \ln\left[\frac{N+R-0{,}5}{N-0{,}5}\right]$$

Step 4: Calculate the lower prediction limit for T_{N+R} from the formula:

$$T_{RL} = t_N \exp\left[\frac{V}{2NG\hat{\beta}F_{0,95}(2(N-1),V')}\right] \qquad \textit{(time terminated)}$$

$$T_{RL} = t_N \exp\left[\frac{V(N-2)}{2N(N-1)G\hat{\beta}F_{0,95}(2(N-1),V')}\right] \qquad \textit{(failure terminated)}$$

and the upper prediction limit for T_{N+R} from the formula:

$$T_{RU} = t_N \exp\left[\frac{VF_{0,95}(V',2(N-1))}{2NG\hat{\beta}}\right] \qquad \textit{(time terminated)}$$

$$T_{RU} = t_N \exp\left[\frac{V(N-2)F_{0,95}(V',2(N-1))}{2N(N-1)G\hat{\beta}}\right] \qquad \textit{(failure terminated)}$$

where the fractiles of the F distribution are given in Table 5 and V' is the rounded integer value of V.

Step 5: The two-sided 90 % prediction interval for T_{N+R} is given by (T_{RL},T_{RU}).

NOTE One-sided 95 % lower and upper limits for T_{N+R} are T_{RL} and T_{RU}, respectively.

7.7 Test for the equality of the shape parameters $\beta_1, \beta_2, ..., \beta_k$

7.7.1 Case 3 – Time data for every relevant failure for two items from different populations

Step 1: Calculate $\hat{\beta}_1$ for item 1 and $\hat{\beta}_2$ for item 2 from step 2 in 7.2.1.

Step 2: Calculate:

$$S_1 = \sum_{i=1}^{N_1-1} \ln\left[\frac{t_{N_1}}{t_{i1}}\right]$$

and

$$S_2 = \sum_{i=1}^{N_2-1} \ln\left[\frac{t_{N_2}}{t_{i2}}\right]$$

Step 3: Calculate:

$$F = \frac{S_1(N_2-1)}{S_2(N_1-1)}$$

Step 4: If

$$\frac{1}{F_{0,95}\left(2(N_2-1),2(N_1-1)\right)} < F < F_{0,95}\left(2(N_1-1),2(N_2-1)\right)$$

where the fractiles of the F distribution are given in Table 5, then the null hypothesis that the β values are the same cannot be rejected at the 10 % significance level. Otherwise, conclude that the shape parameters of the models fitted to the data for the two items are statistically different.

7.7.2 Case 3 – Time data for every relevant failure for three or more items from different populations

Step 1: Calculate $\hat{\beta}_j$ for item j, $j = 1,2,..., k$ from step 2 in 7.2.1.

Step 2: Calculate:

$$S_j = \sum_{i=1}^{N_j-1} \ln\left[\frac{t_{N_j}}{t_{ij}}\right]$$

Step 3: Calculate:

$N = \sum_{j=1}^{k} N_j$ where k denotes the number of items of the same type

Step 4: Calculate:

$$W = 1 + \frac{1}{3(k-1)}\left[\sum_{j=1}^{k}\frac{1}{2(N_j-1)} - \frac{1}{2(N-k)}\right]$$

Step 5: Calculate:

$$Y = 2(N-k)\ln\left[\left(\frac{1}{N-k}\right)\sum_{j=1}^{k} S_j\right] - \sum_{j=1}^{k} 2(N_j-1)\ln\left[\frac{S_j}{N_j-1}\right]$$

Step 6: If

$$\frac{Y}{W} < \chi^2_{0,90}(k-1)$$

where the fractiles of the χ^2 distribution are given in Table 2, then the null hypothesis that the β values are the same cannot be rejected at a 10 % significance level. Otherwise, conclude that the shape parameter of the models fitted to the different items is statistically different.

Table 1 – Critical values for Cramer-von-Mises goodness-of-fit test at 10 % level of significance

M	Critical value of statistic $C^2_{0,90}(M)$
3	0,154
4	0,155
5	0,160
6	0,162
7	0,165
8	0,165
9	0,167
10	0,167
11	0,169
12	0,169
13	0,169
14	0,169
15	0,169
16	0,171
17	0,171
18	0,171
19	0,171
20	0,172
30	0,172
≥60	0,173

NOTE 1 For time terminated tests, $M = N$.

NOTE 2 For failure terminated tests, $M = N–1$.

Table 2 – Fractiles of the Chi-square distribution

Degrees of freedom υ	$\chi^2_{0,05}(\upsilon)$	$\chi^2_{0,90}(\upsilon)$	$\chi^2_{0,95}(\upsilon)$
2	0,10	4,61	5,99
4	0,71	7,78	9,49
6	1,64	10,65	12,59
8	2,73	13,36	15,51
10	3,94	15,98	18,31
12	5,23	18,55	21,03
14	6,57	21,06	23,69
16	7,96	23,54	26,30
18	9,39	25,99	28,87
20	10,85	28,41	31,41
22	12,34	30,81	33,92
24	13,85	33,20	36,42
26	15,38	35,56	38,89
28	16,92	37,92	41,34
30	18,49	40,26	43,77
32	20,09	42,57	46,17
34	21,70	44,88	48,57
36	23,30	47,19	50,96
38	24,91	49,50	53,36
40	26,51	51,81	55,76
42	28,16	54,08	58,11
50	34,76	63,17	67,51
52	36,45	65,42	69,82
60	43,19	74,40	79,08
62	44,90	76,63	81,37
70	51,74	85,53	90,53
72	53,47	87,74	92,80
80	60,39	96,58	101,88
82	62,14	98,78	104,13
90	69,13	107,57	113,15
92	70,89	109,76	115,39
100	77,93	118,50	124,34
102	79,70	120,68	126,57
110	86,79	129,38	135,48
112	88,57	131,56	137,70
120	95,71	140,23	146,57
122	97,49	142,40	148,78
200	168,28	226,02	233,99
z_p	–1,64	+1,28	+1,64

NOTE 1 Linear interpolation of intermediate values is sufficiently accurate.

NOTE 2 For higher values of υ use $\chi_p^2 = \left[\left(z_p + \sqrt{2\upsilon - 1}\right)^2\right]/2$ where z_p is the corresponding fractile of the standard normal distribution.

Table 3 – Multipliers for two-sided 90 % confidence intervals for intensity function for time terminated data

N	*L*	*U*		*N*	*L*	*U*
3	0,175	6,490		21	0,570	1,738
4	0,234	4,460		22	0,578	1,714
5	0,281	3,613		23	0,586	1,692
6	0,320	3,136		24	0,593	1,672
7	0,353	2,826		25	0,600	1,653
8	0,381	2,608		26	0,606	1,635
9	0,406	2,444		27	0,612	1,619
10	0,428	2,317		28	0,618	1,604
11	0,447	2,214		29	0,623	1,590
12	0,464	2,130		30	0,629	1,576
13	0,480	2,060		35	0,652	1,520
14	0,494	1,999		40	0,672	1,477
15	0,508	1,947		45	0,689	1,443
16	0,521	1,902		50	0,703	1,414
17	0,531	1,861		60	0,726	1,369
18	0,543	1,825		70	0,745	1,336
19	0,552	1,793		80	0,759	1,311
20	0,561	1,765		100	0,783	1,273

NOTE 1 For $N > 100$

$$L \cong \frac{N-1}{N}\left(1+1{,}64\sqrt{\frac{1}{2N}}\right)^{-2}$$

$$U \cong \frac{N-1}{N}\left(1-1{,}64\sqrt{\frac{1}{2N}}\right)^{-2}$$

NOTE 2 Linear interpolation of intermediate values is sufficiently accurate.

Table 4 – Multipliers for two-sided 90 % confidence intervals for intensity function for failure terminated data

N	*L*	*U*		*N*	*L*	*U*
3	0,1712	4,746		21	0,6018	1,701
4	0,2587	3,825		22	0,6091	1,680
5	0,3174	3,254		23	0,6160	1,659
6	0,3614	2,892		24	0,6225	1,641
7	0,3962	2,644		25	0,6286	1,623
8	0,4251	2,463		26	0,6344	1,608
9	0,4495	2,324		27	0,6400	1,592
10	0,4706	2,216		28	0,6452	1,578
11	0,4891	2,127		29	0,6503	1,566
12	0,5055	2,053		30	0,6551	1,553
13	0,5203	1,991		35	0,6763	1,501
14	0,5337	1,937		40	0,6937	1,461
15	0,5459	1,891		45	0,7085	1,428
16	0,5571	1,876		50	0,7212	1,401
17	0,5674	1,814		60	0,7422	1,360
18	0,5769	1,781		70	0,7587	1,327
19	0,5857	1,752		80	0,7723	1,303
20	0,5940	1,726		100	0,7938	1,267

NOTE 1 For $N > 100$

$$L \cong \frac{N-2}{N}\left(1+1{,}64\sqrt{\frac{2}{N}}\right)^{-1}$$

$$U \cong \frac{N-2}{N}\left(1-1{,}64\sqrt{\frac{2}{N}}\right)^{-1}$$

NOTE 2 Linear interpolation of intermediate values is sufficiently accurate.

Table 5 – 0,95 fractiles of the *F* distribution

$F_{0,95}(\nu_1\ \nu_2)$	ν_1										
ν_2	**2**	**4**	**6**	**8**	**10**	**20**	**30**	**40**	**60**	**120**	**∞**
2	19,00	19,20	19,30	19,40	19,40	19,40	19,50	19,50	19,50	19,50	19,50
4	6,94	6,39	6,16	6,04	5,96	5,80	5,75	5,72	5,69	5,66	5,63
6	5,14	4,53	4,28	4,15	4,06	3,87	3,81	3,77	3,74	3,70	3,67
8	4,46	3,84	3,58	3,44	3,35	3,15	3,08	3,04	3,01	2,97	2,93
10	4,10	3,48	3,22	3,07	2,98	2,77	2,70	2,66	2,62	2,58	2,54
12	3,89	3,26	3,00	2,85	2,75	2,54	2,47	2,43	2,38	2,34	2,30
14	3,74	3,11	2,85	2,70	2,60	2,39	2,31	2,27	2,22	2,18	2,13
16	3,63	3,01	2,74	2,59	2,49	2,28	2,19	2,15	2,11	2,06	2,01
18	3,55	2,93	2,66	2,51	2,41	2,19	2,11	2,06	2,02	1,97	1,92
20	3,49	2,87	2,60	2,45	2,35	2,12	2,04	1,99	1,95	1,90	1,84
30	3,32	2,69	2,42	2,27	2,16	1,93	1,84	1,79	1,74	1,68	1,62
40	3,23	2,61	2,34	2,18	2,08	1,84	1,74	1,69	1,64	1,58	1,51
60	3,15	2,53	2,25	2,10	1,99	1,75	1,65	1,59	1,53	1,47	1,39
120	3,07	2,45	2,18	2,02	1,91	1,66	1,55	1,49	1,43	1,35	1,25
∞	3,00	2,37	2,10	1,94	1,83	1,57	1,46	1,39	1,32	1,22	1,00
NOTE Linear interpolation for intermediate values is sufficiently accurate.											

Annex A
(informative)

The power law model – Background information

The power law model is widely used to analyse the reliability of repairable items. It is particularly useful for those items classified as 'bad-as-old' when repair is minimal and so reliability of the item remains essentially unchanged after failure and repair. It is also appropriate for those items whose reliability is likely to improve. Indeed the power law model was first considered by L.H. Crow [2][1] in 1974 to describe the power law growth pattern first reported by J.T. Duane in 1964 [5]. Methods for reliability growth analysis based on the power law model are given in IEC 61164 [6].

Crow [2] formulated the underlying probabilistic model for failures as a non-homogeneous Poisson process (NHPP), $\{N(t), t > 0\}$, with an expected value of:

$$E[N(t)] = \lambda t^{\beta}$$

and the failure intensity is given by

$$z(t) = \lambda \beta t^{\beta - 1}$$

The NHPP model gives the Poisson probability that $N(t)$ will assume a particular value, that is:

$$\Pr[N(t) = n] = \frac{(\lambda t^{\beta})^{n} e^{-\lambda t^{\beta}}}{n!} \qquad \text{with } n = 0,1,2,...$$

Also, under this model

$$E[\lambda t_j^{\beta}] = j \qquad \text{with } j = 1,2,...$$

where t_j is the accumulated time to the jth failure. This gives the useful first-order approximation

$$E[t_j] = \left(\frac{j}{\lambda}\right)^{1/\beta} \quad \text{with } j = 1,2,...$$

for the expected time to the jth failure.

When $\beta = 1$, then $z(t) = \lambda$ and the times between successive failures follow an exponential distribution with mean $1/\lambda$ (homogeneous Poisson process), indicating a constant failure intensity. The intensity function $z(t)$ is decreasing for $\beta < 1$ (reliability growth), and increasing for $\beta > 1$ (reliability deterioration).

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1 Figures in square brackets refer to the bibliography.

Annex B
(informative)

Numerical examples

B.1 Background information

The following numerical examples show the use of the procedures discussed in Clause 7. Example 1 considers time data for every relevant failure for a single repairable item when observation is failure terminated. Example 2 considers time data for every relevant failure for multiple repairable items of the same kind when observation is time terminated. Example 3 considers time data for every relevant failure of two repairable items from different populations. Example 4 considers groups of relevant failures for a single repairable item. All examples illustrate the use of appropriate estimation methods. Goodness-of-fit tests are applied when appropriate. These examples may be used to validate computer programs designed to implement the methods given in Clause 7.

Note that all the calculations in the examples were carried out using a spreadsheet software package. Although the final figures are presented to two or three decimal places, the intermediate calculations were carried out in double precision. If intermediate calculations are computed with less precision, then the final figures might differ slightly from those presented due to rounding errors.

It should also be pointed out that all the confidence intervals presented are at a 90 % confidence level and similarly all the statistical tests are conducted at a 10 % significance level. These correspond to the values given in Tables 1 to 5. However, if the appropriate values are taken from corresponding tables reported elsewhere or are generated from software, then alternative values for the confidence and significance levels can be chosen according to users' requirements.

B.2 Example 1

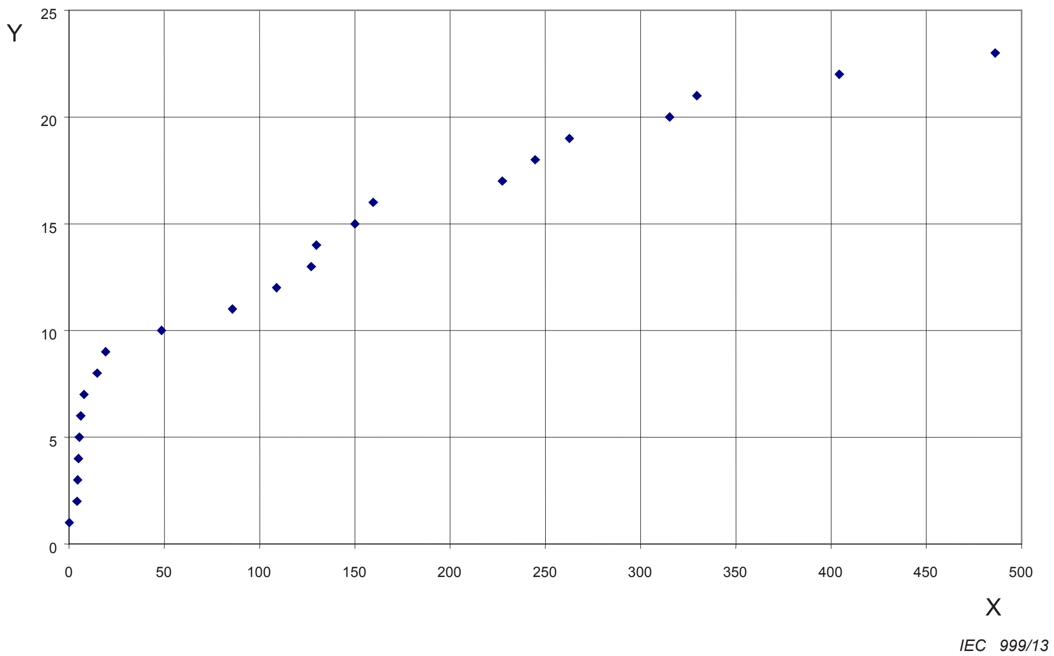
The successive failure times (in hours) of a piece of software developed as part of a large system are given in Table B.1.

Table B.1 – All relevant failures and accumulated times for software system

0,2	4,2	4,5	5	5,4	6,1	7,9	14,8	19,2	48,6	85,8	108,9	127,2
129,8	150,1	159,7	227,4	244,7	262,7	315,3	329,6	404,3,	486,2			
NOTE $t_N = 486{,}2$ h $N = 23$.												

Plot of accumulated failures against time

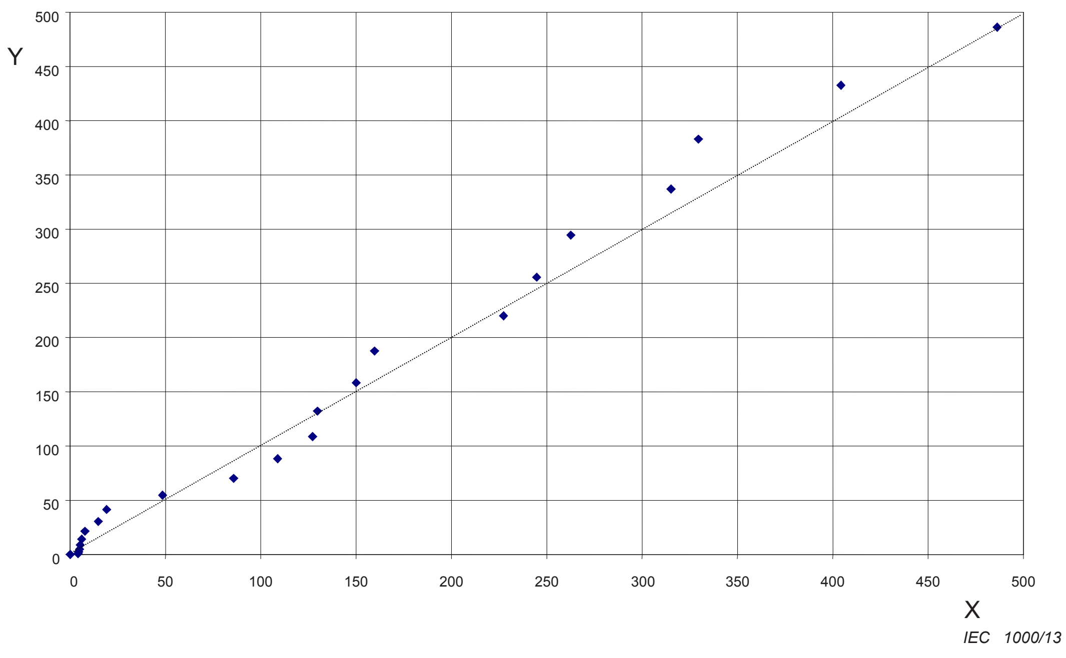
The concave down pattern in Figure B.1 indicates a decreasing failure intensity.



X axis accumulated time (h)

Y axis accumulated number of failures

Figure B.1 – Accumulated number of failures against accumulated time for software system



X axis observed accumulated failure times

Y axis expected accumulated failure times

Figure B.2 – Expected against observed accumulated times to failure for software system

Parameter estimation

From 7.2.1, the estimated parameters of the power law model are as follows:

$$\hat{\lambda} = 2{,}17$$

$$\hat{\beta} = 0{,}38$$

Goodness-of-fit

From 7.3.1.2, the plot of expected against observed failure times in Figure B.2 displays a random scatter around the 45° line indicating a good fit of the power law model to the data. Table B.2 shows the workings for the expected and observed failure times plotted in Figure B.2 where the expected failure times are computed from Step 2 in 7.3.1.2.

From 7.3.1.1, C^2 = 0,063 with M = 22. At a 10 % significance level, the critical value from Table 1 is 0,172. Since 0,063 $<$0,172, it can be concluded that the hypothesis that the power law model is a good fit to the data cannot be rejected.

Table B.2 – Calculation of expected accumulated times to failure for Figure B.2

Failure	**Observed failure time** (h)	**Expected failure time** (h)
1	0,2	0,130
2	4,2	0,803
3	4,5	2,326
4	5,0	4,946
5	5,4	8,881
6	6,1	14,326
7	7,9	21,465
8	14,8	30,468
9	19,2	41,496
10	48,6	54,705
11	85,8	70,242
12	108,9	88,250
13	127,2	108,866
14	129,8	132,224
15	150,1	158,454
16	159,7	187,681
17	227,4	220,028
18	244,7	255,617
19	262,7	294,564
20	315,3	336,983
21	329,6	382,989
22	404,3	432,692
23	486,2	486,200

Confidence interval for β

From 7.4.1, a two-sided 90 % confidence interval for β is (0,27; 0,55). Since all values in this interval are less than one, it indicates a decreasing failure intensity.

Confidence interval for failure intensity

From 7.5.1, a two-sided 90 % confidence interval for the failure intensity at t = 450 h is (0,011; 0,031) failures/h.

Prediction interval for the time to future failures

From 7.6.1, a two-sided 90 % prediction interval for the failure time of the 24th failure is (488,93; 690,30) h. From 7.6.2, a two-sided 90 % prediction interval for the time to the 25th failure is (504,68; 845,30) h.

B.3 Example 2

Five copies of a system were put into operation at the same time under identical conditions. When a system failed it was repaired immediately and returned to operation. The repair time is insignificant compared with the time in operation. Each copy of the system was observed for 1 850 h of operation. The accumulated times to failure are given in Table B.3.

Table B.3 – Accumulated times for all relevant failures for five copies of a system (labelled A, B, C, D, E)

A	B	C	D	E
96	552	1 056	1 560	
1 224	1 225			
1 392	1 570			

The data to be analysed consist of the superimposition of the failure times, which are presented in Table B.4, i.e. the accumulated times for all systems are combined into one data set and presented in their order of occurrence from smallest to largest.

Table B.4 – Combined accumulated times for multiple items of the same kind of a system

Failure	Accumulated time (h)
1	96
2	552
3	1 056
4	1 224
5	1 225
6	1 392
7	1 560
8	1 570

Plot of accumulated failures against time

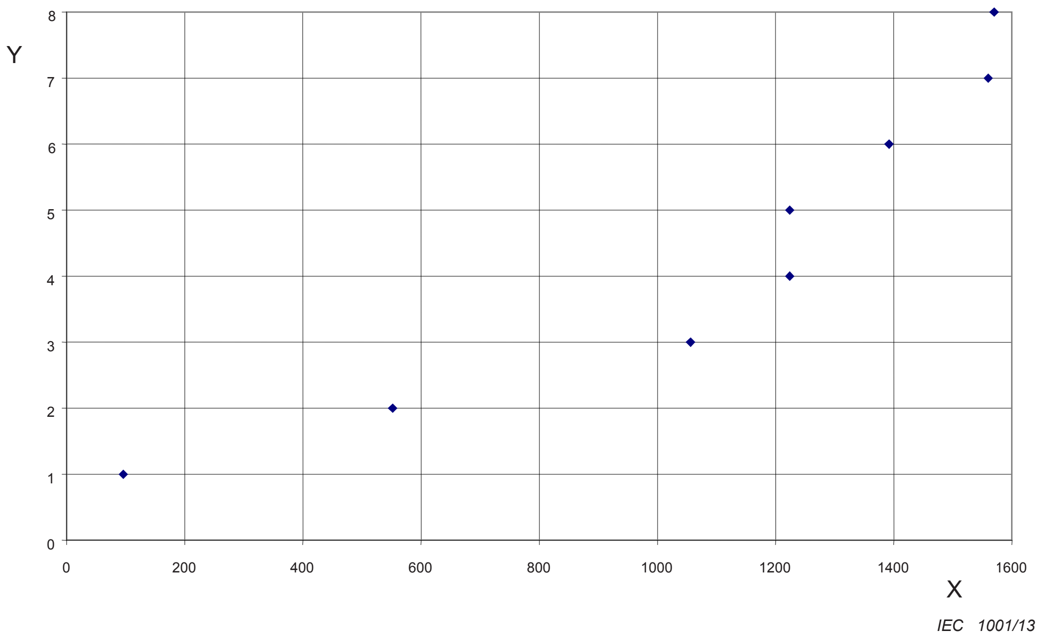
The concave up pattern in Figure B.3 indicates a possible increasing failure intensity.

Parameter estimation

From 7.2.1, the estimated parameters of the power law model are as follows:

$$\hat{\lambda} = 3{,}16 \times 10^{-4}$$

$$\hat{\beta} = 1{,}13$$



X axis accumulated time to failure (h)

Y axis accumulated number of failures

Figure B.3 – Accumulated number of failures against accumulated time for five copies of a system

Goodness-of-fit

From 7.3.1.1, C^2 = 0,115 with M = 8. At a 10 % significance level, the critical value from Table 1 is 0,165. Since 0,115 < 0,165 it can be concluded that the hypothesis that the power law model is a good fit to the data cannot be rejected. This result contradicts the subjective impression stated above. It implies that with only eight failures there is insufficient evidence to discount the power law model. In addition, the confidence intervals given below that are calculated based on this model should be interpreted with the usual caution for such a small data set.

Confidence interval for β

From 7.4.1, a two-sided 90 % confidence interval for β is (0,64; 2,13). Since this interval contains the value 1, it can be concluded that there is no statistical evidence to suggest that the failure intensity is not constant.

Confidence interval for failure intensity

From 7.5.1, a two-sided 90 % confidence interval for the failure intensity at t = 1 000 h is ($3{,}46 \times 10^{-4}$; $23{,}70 \times 10^{-4}$) failures/h.

B.4 Example 3

A manufacturer has tested an OEM (original equipment manufacturer) product from two potential vendors, labelled A and B. After each failure, the units were immediately repaired and returned to test. The accumulated times to failure are given in Table B.5.

Table B.5 – Accumulated operating hours to failure for OEM product from vendors A and B

Accumulated operating hours to failure (Vendor A)	Accumulated operating hours to failure (Vendor B)
600	400
1 100	650
1 500	900
1 750	1 100
2 000	1 500
2 500	2 100
3 100	2 700
3 500	
3 800	
4 500	
NOTE $t_N = 4\,500\,\text{h},\ N = 10$.	NOTE $t_N = 2\,700\,\text{h},\ N = 7$.

Plot of accumulated failures against time

The patterns displayed in Figure B.4 indicate that the failure intensity of both products appears constant, although B has a slightly higher intensity of failure than A.

Parameter estimation

From 7.2.1, the estimated parameters of the power law model are for A

$$\hat{\lambda} = 1{,}53 \times 10^{-3}$$

$$\hat{\beta} = 1{,}04$$

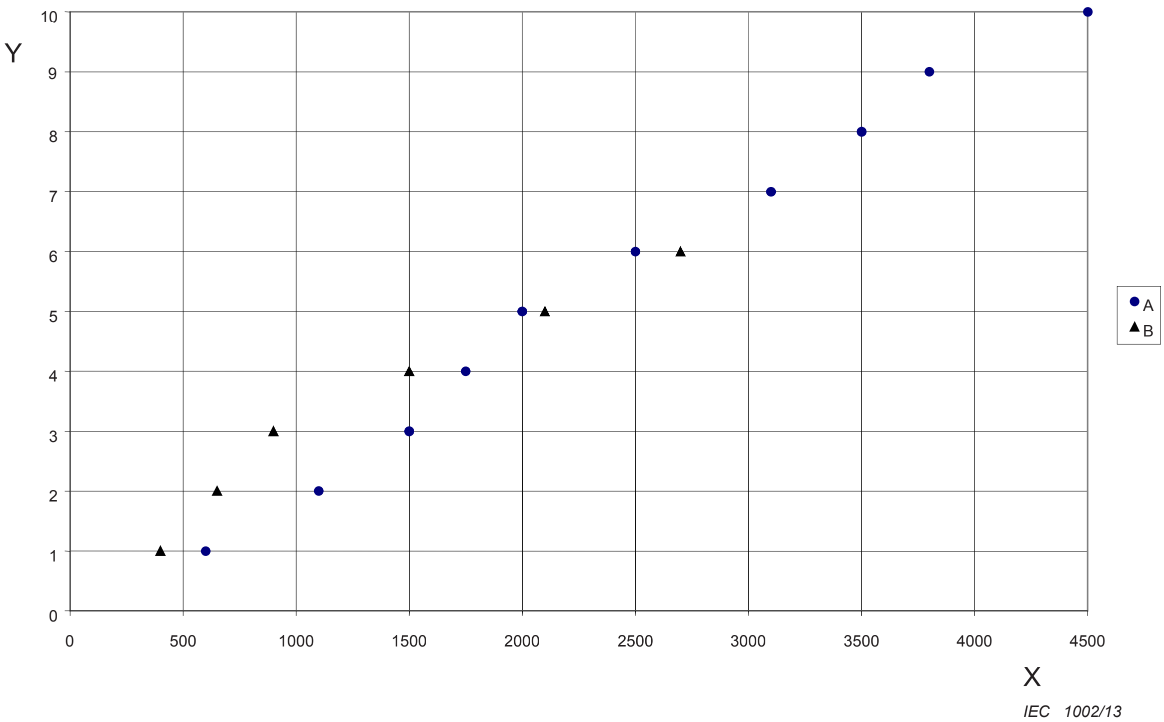
and the estimated parameters of the power law model are for B

$$\hat{\lambda} = 11{,}59 \times 10^{-3}$$

$$\hat{\beta} = 0{,}81$$

Goodness-of-fit

From 7.3.1.1, for A, C^2 = 0,047 with M = 9. At a 10 % significance level, the critical value from Table 1 is 0,167. Since 0,047 $<$ 0,167, one can conclude that the power law model is a good fit to the data. For B, C^2 = 0,072 with M = 6. At a 10 % significance level, the critical value from Table 1 is 0,162. Since 0,072 $<$ 0,162, one can conclude that the hypothesis that the power law model is a good fit to the data cannot be rejected.



X axis accumulated time (h)

Y axis accumulated number of failures

Figure B.4 – Accumulated number of failures against accumulated time for an OEM product from vendors A and B

Confidence interval for β

From 7.4.1, for A, a two-sided 90 % confidence interval for β is (0,61; 1,88) and for B, a two-sided 90 % confidence interval for β is (0,42; 1,70). Since both intervals contain the value 1, conclude that there is no statistical evidence to suggest that both failure intensities are not constant. Since these intervals overlap, there is no evidence to suggest any difference between the constant failure intensities of the two vendors.

Confidence interval for failure intensity

From 7.5.1, for A, a two-sided 90 % confidence interval for the failure intensity at t = 2 500 h is (1,02 × 10^{-3}; 4,80 × 10^{-3}) failures/h. For B, a two-sided 90 % confidence interval for the failure intensity at t = 2 500 h is (0,81 × 10^{-3}; 5,38 × 10^{-3}) failures/h. Since these intervals overlap, there is no evidence to suggest any difference between the failure intensities of the two vendors.

Test of the equivalence of the shape parameters

From 7.7.1 F = 0,83 and from Table 5 $1/F_{0,95}(12,18) = 0,43$ and $F_{0,95}(18,12) = 2,58$. Since 0,43 < 0,83 < 2,58, one can conclude that there is no statistical difference between the shape parameters for the two vendors at a 10 % significance level.

B.5 Example 4

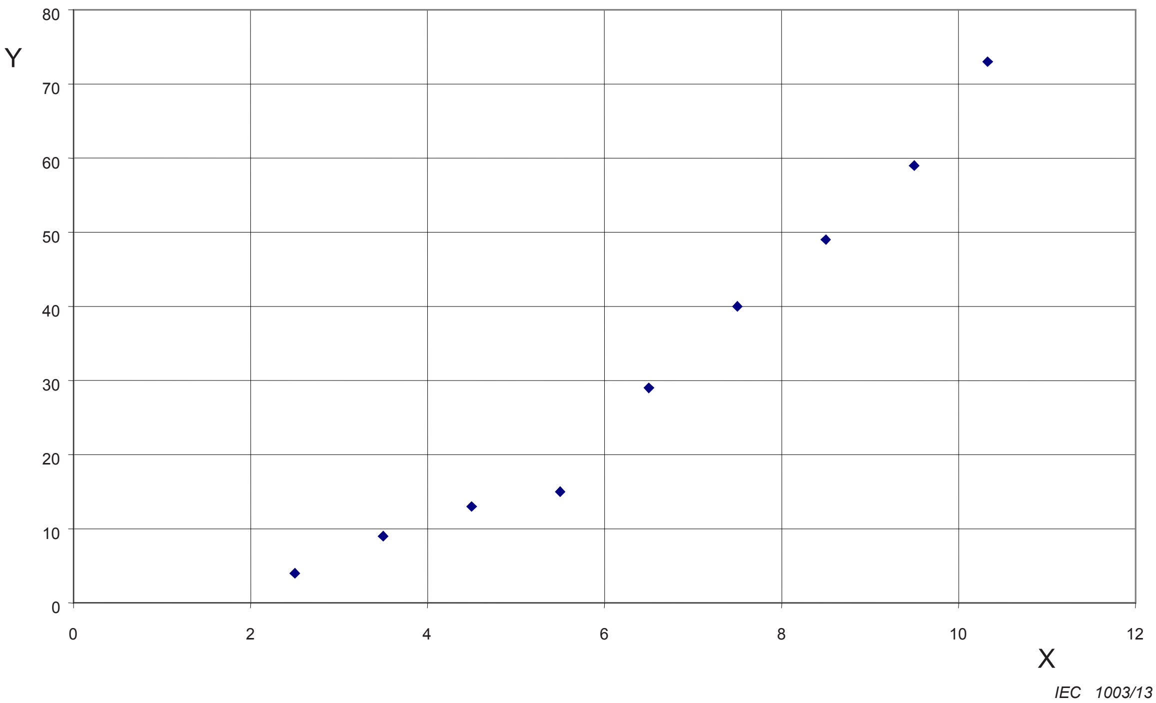
The numbers of failures for generators on a marine vessel are given in Table B.6. The data failures have been recorded in 9 intervals, so set d = 9-1=8.

Table B.6 – Grouped failure data for generators

Accumulated relevant operating time at end of group interval (years)	Number of failures	Accumulated number of failures
0,0	0	0
2,5	4	4
3,5	5	9
4,5	4	13
5,5	2	15
6,5	14	29
7,5	11	40
8,5	9	49
9,5	10	59
10,33	14	73

Plot of accumulated failures against time

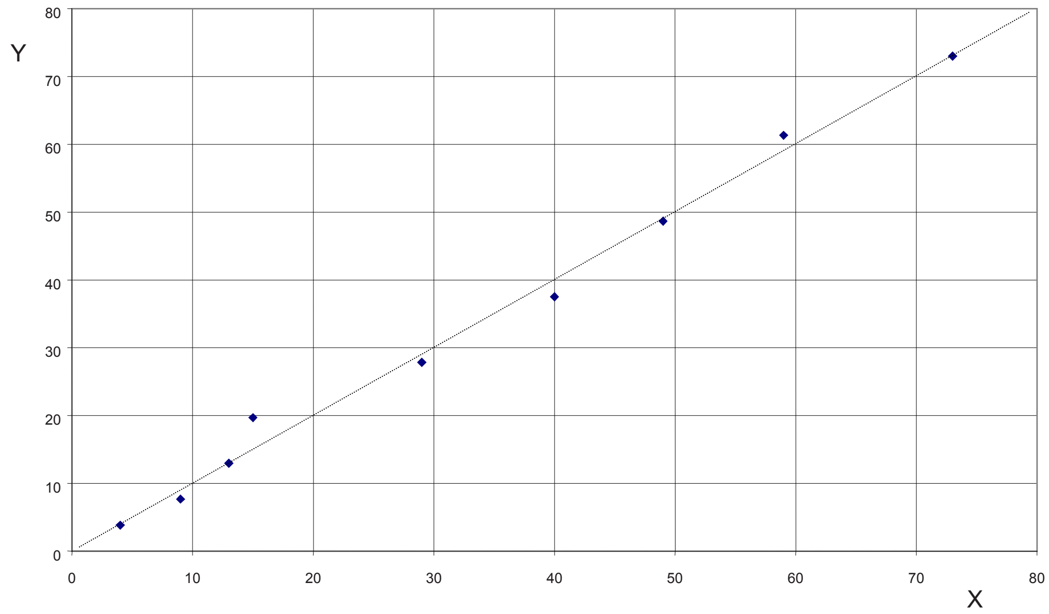
The concave up pattern in Figure B.5 indicates an increasing failure intensity.



X axis accumulated time (years)

Y axis accumulated number of failures

Figure B.5 – Accumulated number of failures against time for generators



X axis observed accumulated number of failures

Y axis expected accumulated number of failures

Figure B.6 – Expected against observed accumulated number of failures for generators

Parameter estimation

From 7.2.3, the estimated parameters of the power law model are as follows:

$$\hat{\lambda} = 0{,}57$$

$$\hat{\beta} = 2{,}08$$

Goodness-of-fit

From 7.3.2.2, the plot of expected against observed accumulated number of failures in Figure B.6 displays a random scatter around the 45° line indicating a good fit of the power law model to the data. Table B.7 shows the workings for the expected and observed accumulated numbers of failures plotted in Figure B.6. The expected number of failures are computed from Step 2 in 7.3.2.2.

Table B.7 – Calculation of expected numbers of failures for Figure B.6

Accumulated relevant operating time at end of group interval (years)	**Observed accumulated number of failures**	**Expected accumulated number of failures**
2,5	4	4,52
3,5	9	7,04
4,5	13	12,12
5,5	15	18,7
6,5	29	26,83
7,5	40	36,55
8,5	49	47,91
9,5	59	60,92
10,33	73	73,00

From 7.3.2.1 the test statistic is $\chi^2 = 9,62$. The critical value from Table 2 is $\chi^2_{0,90}(6) = 10,65$. Since the test statistic is less than the critical value, one can conclude that the power law model is a good fit to the data at a 10 % significance level.

Confidence interval for β

From 7.4.2, a two-sided 90 % confidence interval for β is (1,67; 2,49). Since all values in this interval are bigger than 1, it indicates an increasing failure intensity.

Estimate of failure intensity

From 7.5.2, at t = 11 years, the failure intensity is estimated to be 15,74 failures/year and a 90 % confidence interval is (12,34; 21,74) failures/year.

Annex C
(informative)

Bayesian estimation for the power law model

C.1 Background information

The methods reported in the main body of this standard are based upon a classical approach to statistical estimation. This means that the parameters of the power law model, λ and β, are assumed to be fixed, but unknown, and a classical method, such as 'maximum likelihood', is used to estimate the values of the two parameters using observed data for the accumulated times to failure of a repairable item.

An alternative approach is Bayesian estimation. A Bayesian approach treats the parameters of the power law model, λ and β, as unobserved random variables. This has implications for stages in the estimation process. A Bayesian approach to estimation for the power law process can be summarized in the following stages:

a) choose a probability distribution to reflect the state of knowledge in each of the parameters, λ and β, before collecting any data. This is called the prior distribution;
b) collect observed data for the accumulated times to failure for the repairable item of interest;
c) estimate the parameters of the power law model from the posterior distribution which is computed using Bayes Theorem and reflects what is known about the parameters after observing the data.

The posterior distribution will be proportional to the product of the prior beliefs about the parameters and the so-called likelihood function, which represents the chance of the observed time to failure data being generated from the assumed power law model. In general the posterior can be expressed as follows:

$$posterior \propto likelihood \times prior$$

Table C.1 summarizes the acknowledged strengths and weaknesses of Bayesian estimation compared with classical estimation. The main practical concern of Bayesian estimation relates to the choice of prior. Since the prior will influence the values of estimates obtained, there is a need to clearly state the justification for the form of the prior and to ensure that it is specified before observing data, hence preserving the integrity of the analysis. Otherwise there is a serious risk that the prior may be manipulated to provide estimates that are desired, even if they are not consistent with the observed data. It is recommended that an independent analyst designs and implements an appropriate process to capture and specify the prior distribution from relevant engineering experts with the same rigour as would be applied to collecting observed failure data from the test or field.

The mathematical form of the posterior is related to the distribution function selected for the prior, and in turn this has implications for the complexity of the computations required to obtain the estimates. For classical estimation, there will only be one maximum likelihood estimator and so there is only a need for a single calculation procedure to estimate a parameter, as shown in the main body of this standard. Under Bayesian estimation there will be different formulae and calculation procedures depending upon the form of the prior and posterior distributions. An analyst will be able to give guidance concerning the choice of the type of prior distribution to support both a credible elicitation of engineering judgement and the computation required to obtain the parameters. It is possible that computational software will be required to support Bayesian estimation.

Table C.1 – Strengths and weakness of classical and Bayesian estimation

	Classical	**Bayesian**
Strengths	Well known and accepted by industry Regarded as respecting the objectivity of the data	Existing knowledge can be included Justification for source of relevant information to construct prior for parameters is available
Weaknesses	Assumptions can be hidden To obtain better estimates we require larger sample sizes	Prior is subjective hence there is a risk of selecting a distribution that will influence the results inappropriately Computation can be complex and usually cannot be solved analytically

C.2 Bayesian estimation for the power law model

Consider a power law model with failure intensity given by:

$$z(t) = \lambda \beta t^{\beta - 1}$$

Let $\Pr(\lambda,\beta)$ represent the prior probability for the parameters λ and β. As within the main body of this standard, let t_i represent the accumulated relevant time to the ith failure of a repairable item, where $t_1 < t_2 < \ldots < t_N$. Note that the t_i, $i = 1,\ldots N$, should be observed only after the prior $\Pr(\lambda,\beta)$ has been specified. The posterior distribution will represent the information about the parameters of the model conditional on the observed time to failure data. The posterior distributions is denoted by $\Pr(\lambda,\beta \mid t)$ where $\lambda,\beta \mid t$ denotes the conditional relationship (i.e. denoted by the symbol $\mid$) of the parameters, λ and β, on the times to failure, t_i, $i = 1,\ldots N$. The posterior distribution will be given by:

$$\Pr(\lambda,\beta \mid t) = \frac{\Pr(\lambda,\beta) f(t \mid \lambda,\beta)}{\int_0^\infty \int_0^\infty \Pr(\lambda,\beta) f(t \mid \lambda,\beta) d\lambda d\beta}$$

where $f(t \mid \lambda,\beta)$ is the likelihood function which is given by the joint probability density function of the random variables t_i, $i = 1,\ldots N$, conditional on the parameters λ and β.

Let T represent the accumulated relevant time of a repairable item. Then the likelihood function of the power law process is given by:

$$f(t \mid \lambda,\beta) = \lambda^N \beta^N \left(\prod_{i=1}^{N} t_i \right)^{\beta - 1} e^{-\lambda T^\beta} \qquad \text{(C.1)}$$

NOTE 1 For the case of failure terminated data T is equal to t_N.

The form of the prior distribution should be specified to represent the pattern of uncertainty in the value of the relevant parameter. The functional form of the prior distribution will usually be made by the analyst based upon the problem structure and the information available from relevant engineering experts who may draw upon data for similar systems, test results and other relevant data to inform their subjective judgement.

Many different functional forms for the prior distribution of the power law model exist; Rigdon and Basu [9] provide a review. It is not appropriate to examine all possible prior distributions in this annex. This annex presents two practical examples to illustrate two possible approaches with different forms of the prior distribution and Bayesian estimation calculations for the power law model. By providing details of each step in the modelling of each example problem, it is intended to make the stages of the Bayesian estimation process transparent.

NOTE 2 The two examples given should not be regarded as the only forms of the prior distribution that can be appropriate in practice. Many forms of the prior for the power law model lead to complex calculations that require specialist software or freeware. Advice should be sought from a suitable technical analyst as required.

The prior distribution should be fully specified before observing any data although this time discontinuity in the implementation of the different stages of the Bayesian analysis can be masked within the examples.

C.3 Numerical examples

C.3.1 General

The following numerical examples show how Bayesian estimation for the power law process can be implemented. The code for the calculations was written in computational software. The calculations are shown to four decimal digits. Both examples follow a common format which begins with a description of the background to the problem and then implements the three stages in Bayesian estimation described in Clause C.1.

NOTE The stages of analysis are shown in the sequence that they would be implemented, although it would be good practice for the analyst to fully specify the mathematical model to obtain the posterior distribution and the estimation procedure on selecting the form of the prior.

C.3.2 Bayesian estimation of growth in reliability for a new system in early operation

Background to the problem

A new system has entered service and will be in continuous operation. During early life all hardware faults identified will be addressed through the implementation of appropriate corrective action. Estimates of any changes in the failure intensity are required to assess reliability growth. The power law model provides a credible model for this problem as it can capture changes in the failure intensity as operating experience is accumulated. The engineer responsible for the system has prior knowledge based upon experience of testing and operating similar systems from the same product family.

Stage 1 – Choosing the prior distribution

The analyst prepares a process to capture the engineer's beliefs about the true values of the parameters of the power law model at the point of entry into service of the system. The analyst will consider possible mathematical forms of the prior distribution before implementing an elicitation process to specify the subjective judgement about the engineer's beliefs about the uncertainty in the parameters.

In this case, the analyst decides to re-parameterize the power law model in terms of $\eta = \lambda T^{\beta}$ for the following reasons. First, the new parameter, η, represents the expected number of failures by time T which should be be more meaningful to interpret and support elicitation of engineering judgement. Second, this re-parameterization allows the likelihood function to be expressed as two independent functions which supports elicitation of structured engineering judgement and facilitates computation. The likelihood, previously given in formula (C.1) can be written as:

$$f(t|\eta,\beta) = \prod_{i=1}^{N} t_i^{-1} \left[\beta^N e^{-\beta \sum_{i=1}^{N} \ln \frac{T}{t_i}} \right] \left[\eta^N e^{-\eta} \right] \tag{C.2}$$

where $\Pr(\beta) \sim Gamma\left(N+1, \sum_{i-1}^{N} ln\frac{T}{ti}\right)$ and $\Pr(\eta) \sim Gamma\ (N = 1{,}1)$

NOTE 1 $Gamma(a,b)$ denotes a Gamma distribution with parameters a and b, $\Pr(\beta)$ denotes the probability distribution of β, and $\Pr(\eta)$ is the probability distribution of η.

The analyst requires to select a distribution to represent the prior knowledge about the two parameters β and η. In both cases a Gamma distribution is selected for two reasons. First, it provides a flexible function that should capture the anticipated patterns in the uncertainty in the a priori values of the parameters. Second, the Gamma distribution provides a so-called conjugate prior meaning that the computations to obtain the posterior estimates are more straightforward.

Assume that the parameters β and η are statistically independent and the uncertainty in their true values can be represented by the Gamma prior distributions given by, respectively:

$$\pi(\beta) \sim Gamma\ (a_\beta, b_\beta) \text{ and } \pi(\eta) \sim Gamma\ (a_\eta, b_\eta) \quad \text{(C.3)}$$

We can obtain the values of the so-called hyperparameters, $(a_\beta, b_\beta, a_\eta, b_\eta)$ and check the appropriateness of the Gamma distribution as a representation of the pattern in the uncertainty about β and η through a structured elicitation of engineering judgement. Then we can re-express our prior distributions in terms of our original parameters of the power law model, λ and β as a result of the following relationship:

$$\pi(\eta,\beta) = \pi(\eta)\pi(\beta) = \pi(\lambda \mid \beta)\pi(\beta) = \pi(\lambda,\beta) \quad \text{(C.4)}$$

where $\pi(\lambda) \sim Gamma\ (a_\eta, b_\eta T^\beta)$ and the joint prior distribution $\pi(\lambda,\beta)$ is conjugate.

One approach to capturing the uncertainty in β is to prepare a grid as shown in Table C.2a. The engineering expert is asked to allocate 20 tokens, each worth 5 %, into the different classes within the grid to reflect the chance of the value of the rate of growth β being within a particular class. Table C.2b shows a completed grid where the engineer has indicated that the uncertainty in the true value of β lies between 0 and 0,6 with the modal class being 0,3 – 0,4.

NOTE 2 The engineer is briefed that there is no correct answer to the elicitation question and so an honest opinion about the uncertainty in the possible values of the parameters should be provided.

NOTE 3 The number of tokens, and hence their worth, are chosen to reflect the partitioning of the prior distribution. For example, the total distribution is worth 100 %, hence if it is split into 5 % tokens then 20 are required. If the percentage allocation is reduced (increased) then the number of tokens will increase (decrease) respectively.

NOTE 4 In this example, possible values of the shape parameter are pre-specified on the grid. These can be left blank if it is believed this may cause some anchoring on the classes specified by the analyst.

A similar process can be used to elicit the possible values of the expected number of failures by a specified time T. Table C.3a shows a blank grid for the parameter, η. First, the engineering expert requires to determine meaningful classes for the range of values of η for the case when the system has been in service for 2 years when it is expected to have accumulated 20 000 hours of operational experience, i.e. $T = 20\ 000\ \text{h}$. Tokens can be allocated to classes in accordance with the expert's belief that the true value may fall within each of the classes on the grid. Table C.3b shows a completed grid. The expert believes the true value of the expected number of failures by 20 000 operating hours may lie between 10 000 and 100 000, with the modal class being 30 000 failures.

The analyst shall convert the subjective frequency distributions represented in the grids in Tables C.2 and C.3 into a parametric prior probability distribution. To obtain the prior distributions in Formula (C.1), the analyst shall fit appropriate Gamma distributions to the subjective distributions elicited from the engineering expert. Standard distribution fitting algorithms can be used to find a suitable Gamma distribution for each of the subjective distributions for β and η. A Gamma distribution with parameters $a_\beta = 6,7956$ and $b_\beta = 1/0,0448 = 22,3214$ is found to represent the subjective distribution for the shape parameter β. The best fit for the subjective distribution for the parameter representing the expected number of failures by time $T = 20\ 000\ \text{h}$, η, is a Gamma distribution with parameters $a_\eta = 17,7566$ and $b_\eta = 1/1447,408 = 0,000691$.

Checks on the credibility and the statistical fit of these Gamma distributions should be undertaken. Figures C.1 and C.2 show the plots of the two Gamma distributions and these should be shown to the engineering expert to ensure that the characteristics of the function used to summarize the expressed uncertainties are acceptable. If not, then the analyst should revisit the fitting process to ensure that the probability distribution selected does capture the subjective beliefs of the engineer.

NOTE 5 This can be done by simulating different outcomes of the test (e.g. zero failures, few failures or many failures) and presenting these results to the engineering expert.

Tables C.4 and C.5 show the comparison between the values of the fitted Gamma and the elicited subjective probabilities. The match is not perfect because the fitted Gamma distributions both underestimate the modal class of the subjective distribution by ensuring a better fit in the distribution tails. To better capture the engineer's uncertainties in the distribution tail rather than match the mode of the distribution is a conservative strategy when selecting a parametric prior. Summary statistics, such as the mean of the absolute error of the fitted relative to the subjective probabilities and the standard deviation of the error, can be computed. The analyst will be able to use such summaries to compare fits between competing probability distributions and to assess whether the error is acceptable. In this example, the mean absolute error is of the order of 0,05 which is considered tolerable.

Table C.2 – Grid for eliciting subjective distribution for shape parameter β

Table C.2a – Blank grid pre-elicitation

Possible values of β	(0 – 0,2)	(0,2 – 0,3)	(0,3 – 0,4)	(0,4 – 0,6)	>0,6

Table C.2b – Completed grid post elicitation

			●		
			●		
		●	●		
		●	●		
		●	●		
	●	●	●	●	
	●	●	●	●	
	●	●	●	●	
Possible values of β	(0 – 0,2)	(0,2 – 0,3)	(0,3 – 0,4)	(0,4 – 0,6)	>0,6

Table C.3 – Grid for eliciting subjective distribution for expected number of failures parameter η

Table C.3a – Blank grid pre-elicitation

Possible values of $\eta(\times 10^4)$	0	1	2	$2\frac{1}{2}$	3	5	8	$2\frac{1}{2}$

Table C.3b – Completed grid post elicitation

					●			
					●			
				●	●			
			●	●	●	●		
		●	●	●	●	●		
		●	●	●	●	●	●	●
Possible values of $\eta(\times 10^4)$	0	1	2	$2\frac{1}{2}$	3	5	8	10

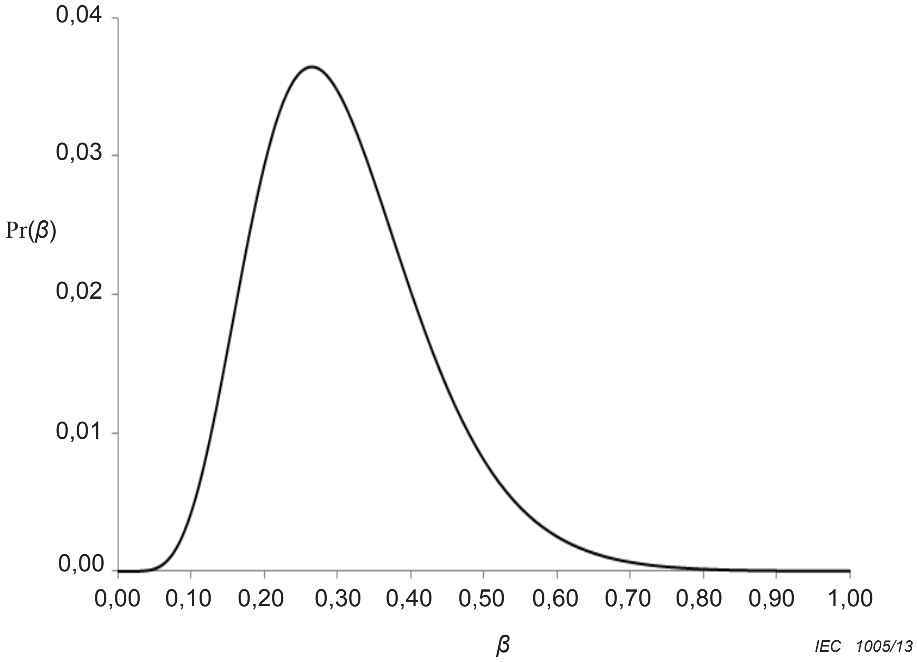


Figure C.1 – Plot of fitted Gamma prior (6,7956, 0,0448) for the shape parameter of the power law model

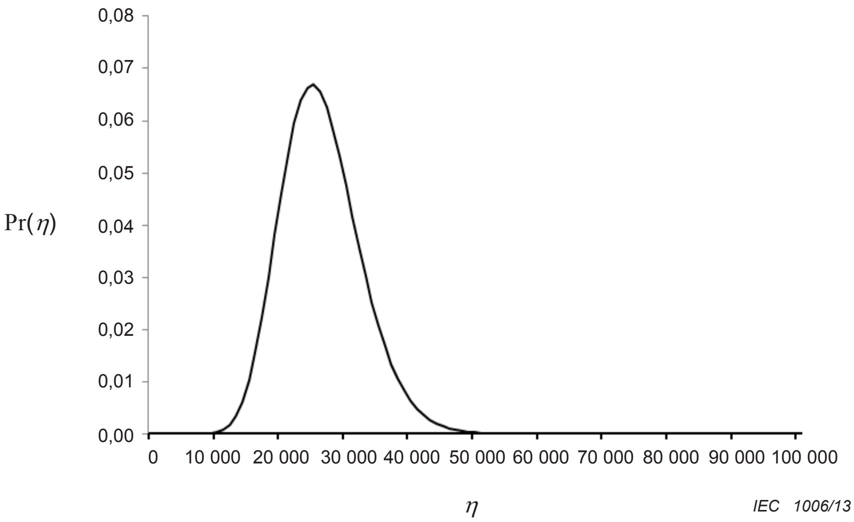


Figure C.2 – Plot of fitted Gamma prior (17,756 6, 1447,408) for the expected number of failures parameter of the power law model

Table C.4 – Comparison of fitted Gamma and subjective distribution for shape parameter β

Interval for possible value of β	Subjective frequency distribution	Subjective probability distribution	Fitted Gamma (6,7956, 0,0448) probability distribution	Error between fitted Gamma relative to subjective probability
0 – 0,2	3	0,15	0,1853	-0,0353
0,2 – 0,3	6	0,30	0,3488	-0,0488
0,3 – 0,4	8	0,40	0,2726	0,1274
0,4 – 0,6	3	0,15	0,1758	-0,0258
>0,6	0	0	0,0175	-0,0175
			Mean absolute error SD of error	-0,0501 0,0722

Table C.5 – Comparison of fitted Gamma and subjective distribution for expected number of failures by time $T = 20\ 000\ \text{h}$ parameter η

Interval for possible values of η	Subjective frequency distribution	Subjective probability distribution	Fitted Gamma (17,7566 , 1447,408) probability distribution	Error between fitted Gamma relative to subjective probability
0 – 10000	2	0,10	0,0004	0,0996
10000 – 20000	3	0,15	0,1748	-0,0248
20000 – 25000	4	0,20	0,3103	-0,1102
25000 – 30000	6	0,30	0,2871	0,0129
30000 – 50000	3	0,15	0,2269	-0,0769
50000 – 80000	1	0,05	0,0006	0,0494
80000 – 100000	1	0,05	0	0,0500
			Mean absolute error SD of error	0,06065 0,0749

Table C.6 – Times to failure data collected on system test

Component description	Failures	Accumulated operating time to failure h
A	4	34h 187 6h 111 43h 12 429h
B	2	10 910h 12 241h
C	1	1 719h
D	3	798h 163 4h 2 692h
E	1	156h
F	2	384h 1 078h
G	1	415h
H	2	11 785h 20 200h
I	5	1h 32h 2 878h 15 973h 18 840h
J	1	1h
K	1	1 235h
L	1	8 286h
M	2	862h 2 074h
N	5	158, 546h 2 828h 2 971h 12 961h
O	3	4 102h 6 523h 13 576h
P	1	15h 178h
Q	4	700h 1 647h 4 121h 12 464h
R	5	18h 45h 575h 611h 13 994h
S	8	5h 11h 226h 1 991h 3 089h 3 989h 5 589h 16 850h

Stage 2 – Observed data for the accumulated times to failure

Table C.6 shows the accumulated operating hours to relevant failures for each system hardware component for which a corrective action was implemented for the system during the first two years of operation. During operation the system accumulated $T = 20\ 000$ h.

Stage 3 – Bayesian estimates of the parameters from the posterior distribution

The observed data can be combined with the prior distribution to generate the posterior distribution from which Bayesian estimates of the power law parameters can be obtained. For the power law model with likelihood function given by formula (C.1) and form of the prior distribution given in formula (C.4), the posterior distribution is given by:

$$\Pr\left(\lambda,\beta|t\right) = Gamma\left(a_\beta + N, b_\beta + \sum_{i=1}^{N}\ln\frac{T}{t_i}\right) \times Gamma\left(a_\eta + N, b_\eta + 1\right) \quad \text{(C.5)}$$

From the observed data in Table C.6, then $N = 52$ failures. Matching the estimated parameters of the fitted Gamma distributions to formula (C.5) gives the following values for the parameters of the posterior distributions:

$$a_\beta + N = 6{,}7956 + 52 = 58{,}7956$$

$$b_\beta + \sum_{i=1}^{N} \ln \frac{T}{t_i} = 22,2816 + 146,4683 = 168,7499$$

$$a_\eta + N = 17,7566 + 52 = 69,7566$$

$$b_\eta + 1 = 0,000691 + 1 = 1,000691$$

The Bayes estimate of the shape parameter β is given by:

$$\hat{\beta} = \frac{a_\beta + N}{b_\beta + \sum_{i=1}^{N} \ln \frac{T}{t_i}} = \frac{58.7956}{168,7499} = 0,3484 \qquad \text{(C.6)}$$

The Bayes estimate of parameter η is given by:

$$\hat{\eta} = \frac{a_\eta + N}{b_\eta + 1} = \frac{69,7566}{1,000691} = 69,7085$$

which yields:

$$\hat{\lambda} = \frac{\eta}{T^\beta} = \frac{69,7085}{20\ 000^{0,3484}} = 2,2117 \qquad \text{(C.7)}$$

Concluding remarks

Table C.7 summarizes the Bayesian and the classical estimates for this example. The workings to obtain the classical estimates are not shown, but use the same steps given within the main body of this standard. Both estimates indicate that the failure intensity of the system is decreasing as operational experience is accumulated and is consistent with reliability growth. The Bayesian estimate of growth is higher than the classical estimate because the Gamma prior distribution for the shape parameter influences the estimated value together with the observations.

The choice of the functional form for the prior, the methods used to elicit and verify the subjective probabilities and the approach used to fit a parametric distribution to the subjective probabilities are important because they impact upon the estimates obtained. In this example, the information in prior distribution influences the Bayesian estimate of the shape parameter. The practical credibility of all assumptions made in the analysis shall be justifiable.

Table C.7 – Summary of estimates of power law model parameters

Parameter	Bayes Estimate	Classical estimate
β	0,3484	0,3467
λ	2,2117	1,6715

C.3.3 Bayesian estimate of future number of failures for an operational system

Background to the problem

An estimate of the number of failures expected during the next 6 000 h once a system has been in operation for 10 000 h is required. A power law model is selected to describe the underlying pattern in the failure intensity as it is believed that this may change through calendar time. Engineering knowledge about the operational demands and planned maintenance will be used to inform the analyst's choice of the prior distribution about the likely number of failures and the associated uncertainty.

Stage 1 – Choosing the prior distribution

The analyst asks the engineering expert to provide judgements about the typical number of failures that he would expect by T = 10 000 h of operation together with an estimate of the spread.

The engineer believes that there may be, on average, 30 failures. However the engineer states that he would be surprised if there were less than 5 or more than 85 failures. Sketching the shape of the distribution of the number of failures, the engineer produces the function shown in Figure C.3.

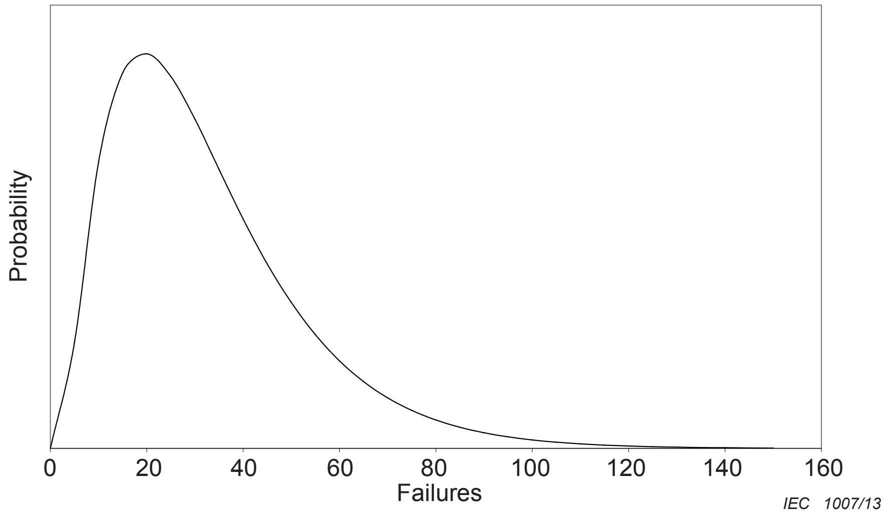


Figure C.3 – Subjective distribution of number of failures

The analyst requires to convert the information about the subjective distribution into a mathematical prior distribution. The analyst aims to select a function that both matches the subjective beliefs of the engineer and facilitates computations for the estimation. The approach adopted is to re-parameterize the power law model to have the intensity function:

$$z(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \qquad \text{(C.8)}$$

where $\theta = \lambda^{-\beta}$ and uses a joint probability for β and θ of the form:

$$\Pr(\theta,\beta;a,b,T) = g(\beta)\frac{\beta}{\theta}\frac{b^a}{\Gamma(a)}\left(\frac{T}{\theta}\right)^{\beta a}\exp\left[-b\left(\frac{T}{\theta}\right)^{\beta}\right] \qquad \text{(C.9)}$$

where $g(\beta)$ is the prior for β and $\Gamma(.)$ is a gamma function. This form of the prior has been proposed by several authors, including Beiser and Rigdon [10] and analysts believe that the

Gamma probability distribution provides a class of models that is sufficiently flexible to capture the pattern in the uncertainty in the number of failures expressed by the engineer.

The hyperparameters of the prior distribution given in formula (C.9), a, b at time T, can be obtained by matching the information provided by the engineer. The analyst can directly equate the expected 30 failures to the mean of the Gamma distribution. The standard deviation gives a measure of spread and for skewed distributions, such as the one shown in Figure C.3, the standard deviation is approximately equal to a quarter of the range. Since the range of failures given by the engineer is 85 – 5 = 80, then the standard deviation can be estimated as 20.

Since the mean and standard deviation of the Gamma distribution can be related to its parameters, the values of a and b can be obtained as follows:

$$a = \frac{\text{mean}^2}{\text{variance}} = \frac{30^2}{20^2} = 2{,}25, \quad b = \frac{\text{mean}}{\text{variance}} = \frac{30}{20^2} = 0{,}075$$

The analyst can generate a plot of the function of a Gamma with parameters (2,25, 0,075) and allow the engineer to verify that this distribution is consistent with the subjective beliefs. If it is not, then the analyst shall revisit the selection of the prior.

In order to fully specify the joint prior distribution given in formula (C.9), the engineer is asked to specify a distribution for the shape parameter β by reasoning through the pattern in the failure intensity. The engineer is confident that the failure intensity will not increase as operational experience is accumulated but has no view as to whether it is more or less likely to decrease or stay constant. The analyst translates this information to a Uniform distribution over the range $0{,}5 < \beta < 1$, giving:

$$g(\beta) = \frac{1}{0{,}5} \quad 0{,}5 < \beta < 1$$

because this function captures the indifference to values of the shape parameter over a range consistent with a non-increasing failure intensity.

Stage 2 – Observed data for the accumulated times to failure

Failure data have been collected from the field. During 10 000 h of operation, N = 30 relevant failures have occurred. The times at which the failures occurred are given in Table C.8.

Stage 3 – Bayesian estimates of the parameters from the posterior distribution

Under the power law with the selected prior distribution, the distribution of the number of failures M in a future time interval $(t_N, t_N + s)$ can be derived and is given by:

$$\Pr(M \mid t) = \frac{cb^a \Gamma(N+M+a)}{M!\Gamma(a)} \int_0^\infty g(\beta) \beta^N T^{\beta a} u^\beta \left\{ \frac{\left[(t_N+s)^\beta - t_n^\beta\right]^M}{\left[bT^\beta + (t_N+s)^\beta\right]^{N+M+a}} \right\} d\beta \quad \text{(C.10)}$$

where N is the number of observed failures at the time of estimation, $u = \prod_{i=1}^{N} t_i$ and c is a normalising constant, given by:

$$c = \left[\int_0^\infty g(\beta) \frac{b^a}{\Gamma(a)} \frac{\beta^N u^\beta T^{a\beta} \Gamma(N+a)}{(t_N^\beta + T^\beta)^{N+\beta}} d\beta \right]^{-1}$$

Substituting the relevant data for the prior and the observed failure data for the system into formula (C.10) gives the posterior distribution for the number of failures M in the future time interval since the last observed failure at 8 690 h, $\left(t_{30} = 8690,\ t_{30} + s = 8690 + 6000\right)$:

$$\Pr\left(M \mid t\right) =$$

$$\frac{c0,075^{2,25}\Gamma\left(30 + M + 2,25\right)}{M!\Gamma\left(2,25\right)} \int_0^\infty \frac{1}{2,5} \beta^{30} 10000^{2,25\beta} u^\beta \left\{ \frac{\left[\left(8690+6000\right)^\beta - 8690^\beta\right]^M}{\left[0,075(10000)^\beta + \left(8690+6000\right)^\beta\right]^{30+2,25+M}} \right\} d\beta$$

where $u = \prod_{i=1}^{N} t_i = 3.7463\text{x}10^{107}$ and $c = \left[3.5887\text{x}10^{-20}\right]^{-1}$.

Table C.8 – Time to failure data for operational system

Failure number	Accumulated time to failure h
1	860
2	1 258
3	1 317
4	1 422
5	1 897
6	2 011
7	2 122
8	2 439
9	3 203
10	3 298
11	3 902
12	3 910
13	4 000
14	4 247
15	4 411
16	4 456
17	4 517
18	4 899
19	4 910
20	5 676
21	5 755
22	6 137

Failure number	Accumulated time to failure h
23	6 211
24	6 311
25	6 613
26	6 975
27	7 335
28	8 158
29	8 498
30	8 690

Figure C.4 shows the posterior probabilities for the number of failures, M, in the next 6 000 h of operation and Figure C.5 shows the cumulative posterior probability distribution for the number of failures in the next 6 000 h of operation. The mean of the posterior distribution is 18,24 which implies that there will most likely be 19 failures in the next 6 000 h of operation. It is also possible to obtain 95 % limits on the number of failures from the posterior distribution. For example, an upper 95 % limit corresponds to the 95 % percentile of the posterior distribution, which has a value of 28. This means that there is a 5 % chance that there will be more than 28 failures in the next 6 000 h of operation.

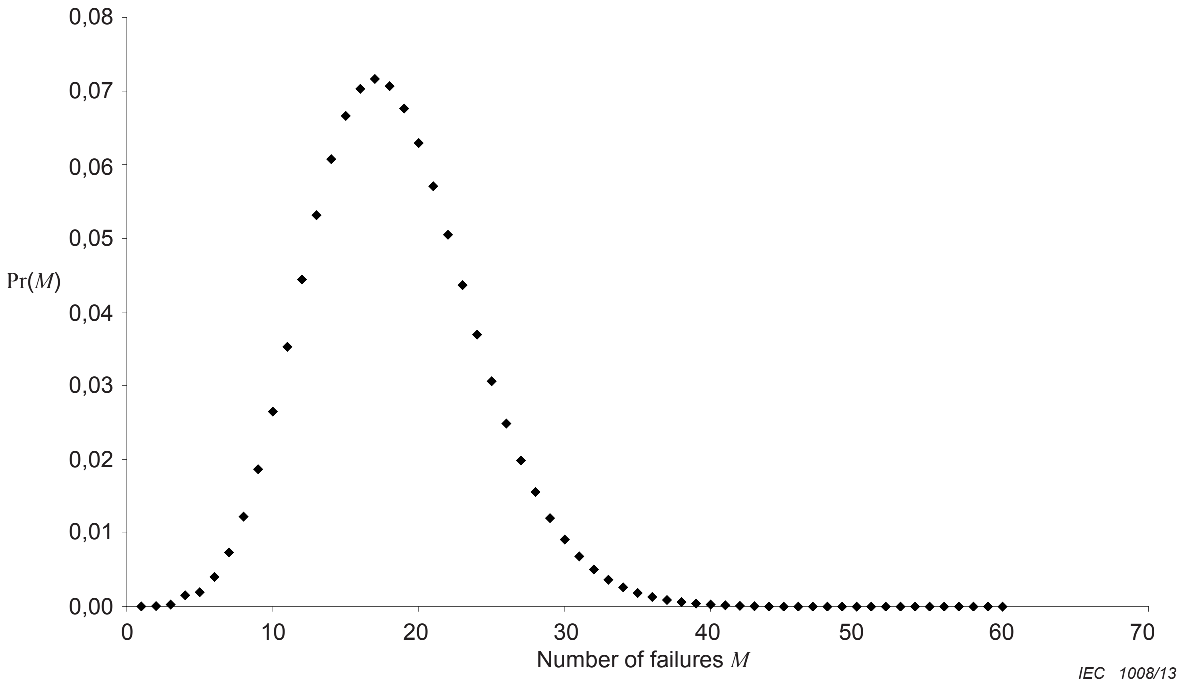


Figure C.4 – Plot of the posterior probability distribution for the number of future failures, M

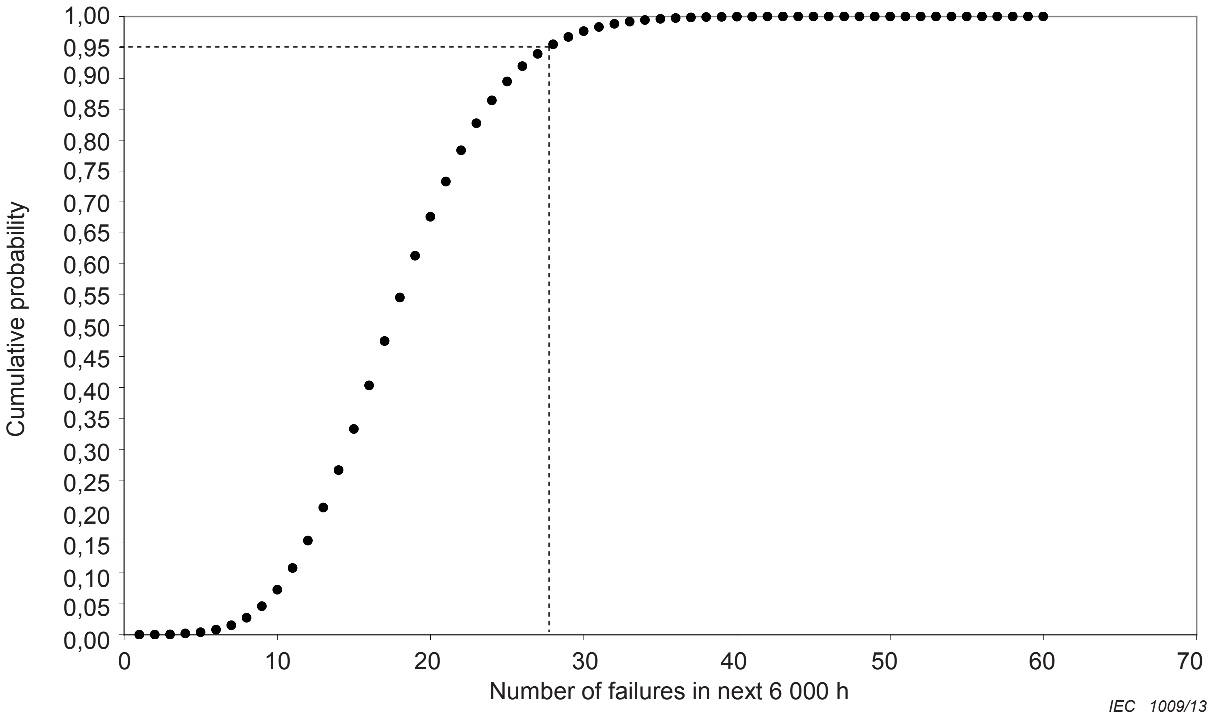


Figure C.5 – Plot of the posterior cumulative distribution for the number of future failures, M

C.4 Summary

The information in this annex aims to explain the rationale of a Bayesian approach to estimation for the power law model. Bayesian estimation allows the analyst to include prior information into the model and to combine this with observed time to failure data. The classical methods, which are explained in the main body of this standard, only use the observed accumulated time to failure data to obtain estimates.

The examples given in this annex show insight into the process of Bayesian analysis for two specific approaches. An analyst who has a sound knowledge of Bayes should be involved in the estimation because Bayesian analysis involves more complex modelling than is usually the case for classical estimation.

Bayesian methods can be very powerful, but consequently should be used with care. In particular, the relevant information used to specify the prior distribution should be fully justified and open to scrutiny to maintain the integrity of the analysis.

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